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Type IIA AdS$_4$ compactifications on cosets, interpolations and domain walls

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Abstract: We present a classification of a large class of type IIA $\mathcal{N} = 1$ supersymmetric compactifications to AdS$_4$, based on left-invariant SU(3)-structures on coset spaces. In the absence of sources the parameter spaces of all cosets leading to a solution contain regions corresponding to nearly-Kähler structure. I.e. all these cosets can be viewed as deformations of nearly-Kähler manifolds. Allowing for (smeared) six-brane/orientifold sources we obtain more possibilities. In the second part of the paper, we use a simple ansatz, which can be applied to all six-dimensional coset manifolds considered here, to construct explicit thick domain wall solutions separating two AdS$_4$ vacua of different radii. We also consider smooth interpolations between AdS$_4 \times \mathcal{M}_6$ and $\mathbb{R}^{1,2} \times \mathcal{M}_7$, where $\mathcal{M}_6$ is a nearly-Kähler manifold and $\mathcal{M}_7$ is the G$_2$-holonomy cone over $\mathcal{M}_6$.

Keywords: Flux compactifications, Superstring Vacua.

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Contents

G	H
G_2	$SU(3)$
$SU(3) \times SU(2)^2$	$SU(3)$
$Sp(2)$	$S(U(2) \times U(1))$
$SU(3) \times U(1)^2$	$S(U(2) \times U(1))$
$SU(2)^3 \times U(1)$	$S(U(2) \times U(1))$
$SU(3)$	$U(1) \times U(1)$
$SU(2)^2 \times U(1)^2$	$U(1) \times U(1)$
$SU(3) \times U(1)$	$SU(2)$
$SU(2)^3$	$SU(2)$
$SU(2)^2 \times U(1)$	$U(1)$
$SU(2)^2$	1

Table 1: All six-dimensional manifolds of the type $M = G/H$, where H is a subgroup of SU(3).

1. Introduction and summary

In recent years, it has become clear that compactifications of string theory in the presence of fluxes can be usefully described in the language of G-structures [1]. In particular the requirement of $\mathcal{N} = 1$ supersymmetry in four dimensions in type II for six-dimensional compactification manifolds of SU(3) structure can be conveniently summarized as a set of necessary conditions on the torsion classes of these manifold [2] (see [3] for a review and further references). It was subsequently realized that generalized geometry [4] provides a natural framework for the most general $\mathcal{N} = 1$ supersymmetric ansatz in type II, also known as SU(3) $\times$ SU(3)-structure, and it was shown in [5] that the supersymmetry conditions can be succinctly rewritten as differential conditions on a pair of polyforms.

A systematic search for concrete examples of six-dimensional manifolds, suitable for $\mathcal{N} = 1$ compactification to four-dimensional Minkowski space, has yielded very few examples [6]. Moreover, due to a no-go theorem [7] these examples require the presence of orientifold planes, typically smeared. In certain cases, it can be argued that the latter arise as the large-volume supergravity approximation of bona-fide string-theory orientifolds.

The situation is somewhat better in $\mathcal{N} = 1$ compactifications to four-dimensional anti-de Sitter space [8, 9] where the no-go theorem can be circumvented. For instance, the six-dimensional compact nearly-Kähler manifolds constitute a viable starting point for supersymmetric compactifications, without the need for orientifolds. Recently, it was pointed out in [10] that the Hopf reductions of eleven-dimensional supergravity considered by Nilsson and Pope [11] lead to supersymmetric IIA compactifications that are not nearly-Kähler, in that the torsion class $\mathcal{W}_2$ is non-zero. Necessarily, however, these solutions have vanishing Romans mass. Subsequently, using twistor-space techniques, the author of [12] constructed compactifications interpolating between the nearly-Kähler and vanishing-Romans-mass cases on two special coset manifolds — each of which can be seen

$\tau \in \mathcal{W}_1^- \oplus \mathcal{W}_2^-$
$d\mathcal{W}_2^- \propto \mathrm{Re}\Omega$
$3\lvert\mathcal{W}_1^-\rvert^2 \geq \lvert\mathcal{W}_2^-\rvert^2$

Table 2: Necessary and sufficient conditions on the internal six-dimensional SU(3)-structure manifold for $\mathcal{N} = 1$ compactification to four-dimensional anti-de Sitter space, in the absence of sources.

as a twistor bundle.[1]

In the present paper we provide a classification of a large class of concrete examples of six-dimensional compact manifolds that satisfy the necessary and sufficient conditions for $\mathcal{N} = 1$ compactification with a strict SU(3)-structure ansatz to four-dimensional anti-de Sitter space. Namely, we consider compactifications on manifolds of the type $M = G/H$, where G is a Lie group (not necessarily simple) and H is a closed subgroup, such that the action of G on M is effective. Coset spaces were studied some time ago, in the context of the Kaluza-Klein approach to unification. For a review see [14] and references therein. For early work, in the context of heterotic string theory, see [15, 16]; for some recent results see [17].

The requirement of four-dimensional supersymmetry imposes the condition that the structure group of $T(M)$, the tangent bundle of the six-dimensional internal manifold M, is reduced to SU(3). As we show in appendix A, this translates into the requirement that H be isomorphic to SU(3) or a subgroup thereof. All possible six-dimensional manifolds M of this type can be easily classified, and consist of the ones listed in table 1, as well as those obtained from the above by replacing any number of SU(2) factors in G by factors of $\mathrm{U}(1)^3$.

As we review in section 2, the necessary and sufficient conditions for $\mathcal{N} = 1$ compactification to four-dimensional anti-de Sitter space on manifolds of SU(3)-structure can be compactly summarized as a set of conditions on the torsion classes of the internal six-dimensional manifold; the resulting geometry is then determined by the fluxes [9]. In particular, the intrinsic torsion, τ, of the six-dimensional manifold must be contained in the first two torsion classes $\mathcal{W}_{1,2}^-$. In the special case where the second torsion class vanishes, $\mathcal{W}_2^- = 0$, the manifold is called *nearly-Kähler.*

In the absence of sources, there are additional constraints on the torsion classes: a) the exterior derivative of the second torsion class must be proportional to the real part of the three-form of the SU(3)-structure, and b) the norm of the first torsion class is bounded below by the norm of the second torsion class. All the conditions are summarized in table 2. Note, however, that in the presence of sources the last two conditions can be relaxed, as we review in the following.

Given the list of table 1, one can systematically search for those manifolds that satisfy the necessary and sufficient conditions for $\mathcal{N} = 1$ compactification to four-dimensional anti-de Sitter space, listed in table 2. As we review in section 3, the coset structure of the manifolds is essential for the analysis, because it allows for the definition of left-invariant

[1] Publication [13] considers the compactification of IIA supergravity on the coset $\mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1)$, but without any analysis of the Bianchi identities of the form-fields.

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	$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$	$\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$	$\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$
# of parameters	1	3	2	1
$\mathcal{W}_2^- \neq 0$	No	Yes	Yes	No

Table 3: Six-dimensional cosets that satisfy the necessary and sufficient conditions for $\mathcal{N} = 1$ compactification to four-dimensional anti-de Sitter space, in the absence of sources.

one-forms on which the action of the exterior derivative is completely determined by the structure constants of the coset. If one further imposes (as we do here) that the SU(3)-structure be *left-invariant*,[2] the torsion classes of the coset (which can be obtained from the SU(3)-structure by exterior differentiation) are completely determined in terms of the structure constants. It then suffices to write down the most general left-invariant ansatz for the SU(3)-structure and impose that the torsion classes satisfy the necessary and sufficient conditions of table 2.

One then ends up with exactly four possibilities, which are listed in table 3. The number of arbitrary parameters (moduli) of each solution is indicated in the first row. More precisely: this is the number of moduli of *left-invariant* SU(3)-structures, such that the conditions of table 2 are satisfied. There is always at least one modulus, corresponding to the overall volume rescaling. Note that although these moduli can be continuous parameters from the point-of-view of classical supergravity, they are determined in terms of the fluxes of the solution (as will be explained in more detail in the following section). Since the fluxes are quantized in the full quantum theory, the 'moduli' can only assume discrete values.

All cosets of table 3 admit points (more precisely: lines) in their moduli spaces which correspond to nearly-Kähler structure (see figure 1). Whenever the moduli space is one-dimensional, i.e. whenever the only modulus is the overall volume, the solution only admits a nearly-Kähler structure. In fact, the list of table 3 is identical to the list of all six-dimensional compact homogeneous manifolds that admit a strictly nearly-Kähler structure [18].[3] On the other hand, whenever there are more parameters than just the volume modulus, i.e. whenever the dimension of moduli space is two or higher, the solution can be deformed away from the nearly-Kähler line.

The second row (labelled by $\mathcal{W}_2^- \neq 0$) indicates whether or not the coset admits a left-invariant SU(3)-structure that is *not* nearly-Kähler. This is indeed the case for the cosets $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ and $\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$, but not for the cosets $\mathrm{SU}(2) \times \mathrm{SU}(2)$, $\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$, which only admit a rigid nearly-Kähler structure. We stress again that *all* cosets of table 3 admit nearly-Kähler structures. In other words, if a coset admits a structure with $\mathcal{W}_2^- \neq 0$ it also admits a structure with $\mathcal{W}_2^- = 0$, but not vice versa.

A number of cosets not listed in table 3 admit solutions which turn out to be equivalent to the ones already listed in the table. More precisely, the cosets $\frac{\mathrm{SU}(2)^2\times\mathrm{U}(1)}{\mathrm{U}(1)}$, $\frac{\mathrm{SU}(2)^3}{\mathrm{SU}(2)}$, $\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)^2}{\mathrm{SU}(3)}$, admit structure constants and left-invariant SU(3) structures which turn out

[2]The restriction to left-invariant SU(3)-structures is made here in order to render the problem tractable. We leave the investigation of more general possibilities for future work.

[3]They are also precisely those coset spaces which were singled out in the first paper in [15].

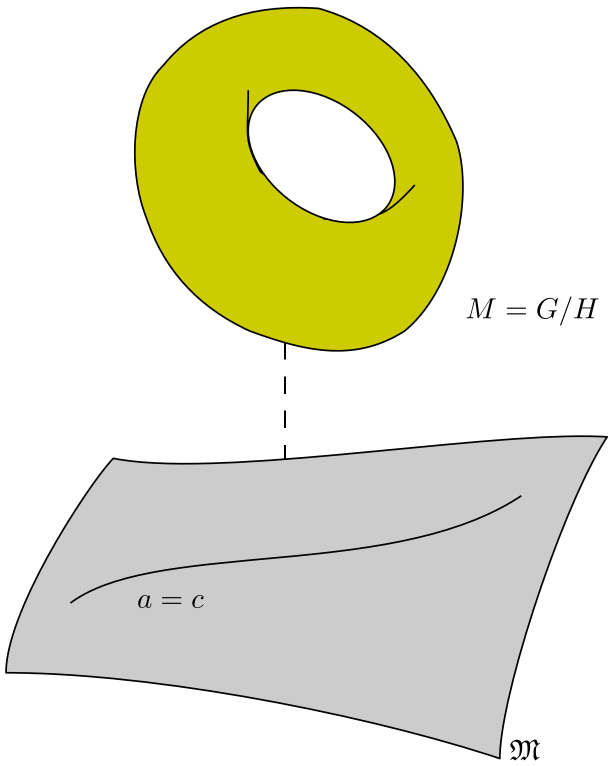

Figure 1: The coset space $\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$ fibered over its two-dimensional moduli space $\mathfrak{M}$. The nearly-Kähler limit corresponds to the line $a = c$ in $\mathfrak{M}$, see section 4.3 below. In the full quantum theory the moduli can only assume discrete values.

to be equivalent to the ones of the SU(2)×SU(2) coset. More details, as well as the structure constants for each coset and the SU(3)-structure for each solution, are given in section 4.

All the cosets listed in table 3 also admit *smeared* six-brane/orientifold sources whose Poincaré dual j^6 is proportional to the real part of the three-form of the SU(3)-structure: $j^6 \propto \mathrm{Re}\Omega$. If one allows for smeared six-brane/orientifold sources that violate this proportionality condition, then there is one additional possibility: $\frac{\mathrm{SU}(3)\times\mathrm{U}(1)}{\mathrm{SU}(2)}$, with the topology of $S^5 \times S^1$. Table 4 lists the cosets that satisfy the necessary and sufficient conditions for $\mathcal{N} = 1$ compactification to four-dimensional anti-de Sitter space, in the presence of smeared sources. The third row indicates whether or not the Poincaré dual of the source is proportional to $\mathrm{Re}\Omega$. In the case of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ there are solutions both with j^6 proportional, and not proportional to $\mathrm{Re}\Omega$.

To conclude the discussion of the coset vacua let us make a remark on possible type IIA/IIB AdS$_4$ supersymmetric backgrounds within the class of coset geometries with more general G-structure than strict SU(3).[4] Obviously for static SU(2), but also for $\mathrm{SU}(3) \times \mathrm{SU}(3)$-structure if one insists on left-invariant structures, supersymmetry requires H to be a subgroup of SU(2). This leaves only the last four entries of table 1 as candidates.

[4]For more details on the meaning of "strict SU(3)", "static SU(2)" and "$\mathrm{SU}(3) \times \mathrm{SU}(3)$" see [5, 19].

	$\mathrm{SU}(2)\times\mathrm{SU}(2)$		$\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$	$\frac{\mathrm{Sp}(2)}{\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$	$\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$	$\frac{\mathrm{SU}(3)\times\mathrm{U}(1)}{\mathrm{SU}(2)}$
# of parameters	2	4	4	3	2	4
$\mathcal{W}_2^- \neq 0$	No	Yes	Yes	Yes	No	Yes
$j^6 \propto \mathrm{Re}\Omega$	Yes	No	Yes	Yes	Yes	No

Table 4: Six-dimensional cosets that satisfy the necessary and sufficient conditions for $\mathcal{N} = 1$ compactification to four-dimensional anti-de Sitter space, in the presence of smeared six-brane/orientifold sources. In our parameter counting we now also include the number of sources, which is of course again a discrete quantity. For $\mathrm{SU}(2)\times\mathrm{SU}(2)$ we distinguish two cases depending on whether or not the source term is proportional to $\mathrm{Re}\Omega$.

We have only found a static SU(2) IIB solution[5] on $\frac{\mathrm{SU}(3)\times\mathrm{U}(1)}{\mathrm{SU}(2)}$, which is T-dual to the IIA SU(3) solution on the same coset, and one on $\frac{\mathrm{SU}(2)^2\times\mathrm{U}(1)}{\mathrm{U}(1)}$, which is T-dual to the IIA SU(3) solution on $\mathrm{SU}(2)\times\mathrm{SU}(2)$.

In the final part of the paper, using a simple ansatz, which can be applied to all six-dimensional coset manifolds $\mathcal{M}_6 = M$ considered here, we have been able to obtain smooth interpolations between two AdS_4 vacua of different radii. These solutions can be interpreted as domain walls in the four noncompact dimensions, and they necessarily contain 'thick' branes. By that we mean branes whose profile in the radial direction (the direction transverse to the wall) is not a delta-function, but is nevertheless localized — in the sense that it falls off to zero far from the wall. However, we have been unable to obtain explicit profiles of non-pathological smooth interpolations between $\mathrm{AdS}_4\times\mathcal{M}_6$ and $\mathbb{R}^{1,2}\times\mathcal{M}_7$, where $\mathcal{M}_7$ is the Hitchin lift of $\mathcal{M}_6$.

2. Review of AdS_4 solutions

The most general form of $\mathcal{N} = 1$ compactifications of IIA supergravity to AdS_4 with the ansatz $\eta^{(1)} \propto \eta^{(2)}$ for the internal supersymmetry generators (the strict SU(3)-structure ansatz) was given by two of the present authors in [9]. These vacua must have constant warp factor and dilaton. Setting the warp factor to one, the solutions of [9] are given by:[6]

$$H = \frac{2m}{5} e^{\Phi} \mathrm{Re}\Omega\,, \tag{2.1a}$$

$$F_2 = \frac{f}{9} J + F_2'\,, \tag{2.1b}$$

$$F_4 = f \mathrm{vol}_4 + \frac{3m}{10} J\wedge J\,, \tag{2.1c}$$

$$W e^{i\phi} = -\frac{1}{5} e^{\Phi} m + \frac{i}{3} e^{\Phi} f\,. \tag{2.1d}$$

[5]Recall that it is impossible to have supersymmetric IIA AdS_4 solutions with static SU(2) structure [20].

[6]As opposed to [9] we do not use superspace conventions. Furthermore we use here the string frame and put $m = -2m_{\text{there}}, H = -H_{\text{there}}, J = -J_{\text{there}}, F_2 = -2m_{\text{there}}B'$ and $F_4 = -G$.

In the above $(J,\,\Omega)$ is the SU(3)-structure of the internal six-manifold, i.e. J is a real two-form and Ω is a complex three form such that:

$$\Omega\wedge J=0\,, \tag{2.2a}$$

$$\Omega\wedge\Omega^*=\frac{4i}{3}J^3\neq 0\;. \tag{2.2b}$$

f, m are constants parameterizing the solution: f is the Freund-Rubin parameter, while m is the mass of Romans' supergravity [21] — which can be identified with F_0 in the 'democratic' formulation [22]. $e^{i\phi}$ is a phase associated to the internal supersymmetry generators $\eta_+^{(2)}=e^{i\phi}\eta_+^{(1)}$. W is defined by the following relation for the AdS Killing spinors

$$\nabla_\mu\zeta_-=\frac{1}{2}W\gamma_\mu\zeta_+\,. \tag{2.3}$$

The radius of AdS$_4$ is given by $|W|^{-1}$. The two-form F_2' is the primitive part of F_2 (i.e. it is in the **8** of SU(3)) and is constrained by the Bianchi identity:

$$\mathrm{d}F_2'=\left(\frac{2}{27}f^2-\frac{2}{5}m^2\right)e^\Phi\mathrm{Re}\Omega-j^6\,, \tag{2.4}$$

where we have added a source for D6-branes/O6-planes on the right-hand side. We immediately see that in the absence of sources the second constraint of table 2 holds, i.e. $d\mathcal{W}_2^-\propto\mathrm{Re}\,\Omega$. However in the presence of nonzero j^6, this constraint may be relaxed.

The general properties of supersymmetric sources, and their consequences for the integrability of the supersymmetry equations, were recently discussed by two of the present authors in [19], within the framework of generalized geometry. It was shown in this reference that, under certain mild assumptions, which can be seen to be satisfied in the present context, the inclusion of a supersymmetric source in the Bianchi identities, which must be generalized calibrated as in [23], does indeed give rise to a new, valid, solution: supersymmetry guarantees that the appropriately source-modified Einstein equation and dilaton equation-of-motion are automatically satisfied.

Finally, the only nonzero torsion classes of the internal manifold are $\mathcal{W}_1^-,\mathcal{W}_2^-$ such that

$$dJ=-\frac{3}{2}i\,\mathcal{W}_1^-\mathrm{Re}\Omega\,, \tag{2.5a}$$

$$d\Omega=\mathcal{W}_1^-J\wedge J+\mathcal{W}_2^-\wedge J\;. \tag{2.5b}$$

Moreover, they are given by:

$$\mathcal{W}_1^-=-\frac{4i}{9}e^\Phi f\,,\qquad \mathcal{W}_2^-=-ie^\Phi F_2'\,. \tag{2.6}$$

For the following it will be convenient to also introduce $c_1:=-\frac{3}{2}i\,\mathcal{W}_1^-$, which appears in (2.5a). In addition, for vanishing sources or for sources proportional to $\mathrm{Re}\Omega$ we can also define c_2 by

$$d\mathcal{W}_2^-=ic_2\,\mathrm{Re}\Omega\,. \tag{2.7}$$

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One can show [9] that

$$c_2 = -\frac{1}{8}|\mathcal{W}_2^-|^2 \,. \tag{2.8}$$

It was further noted in [9] that, for vanishing j^6, the parameters f, m of the solution obey the bound: $f^2 \geq 27/5m^2$, which follows from $|\mathcal{W}_2^-|^2 \geq 0$, (2.4), (2.7) and (2.8), with equality for nearly-Kähler manifolds. However, to determine whether a given geometry $(\mathcal{W}_1^-, \mathcal{W}_2^-)$ corresponds to a vacuum without orientifold sources, the following bound is more relevant

$$\frac{16}{5}e^{2\Phi}m^2 = 3|\mathcal{W}_1^-|^2 - |\mathcal{W}_2^-|^2 \geq 0\,, \tag{2.9}$$

where we have defined $|\Phi|^2 := \Phi^*_{mn}\Phi^{mn}$, for any two-form Φ. Incidentally, let us note that condition (2.9) turns out to be too stringent to be satisfied for any nilmanifold whose only nonzero torsion classes are $\mathcal{W}_{1,2}^-$ [24].

Allowing, however, for a nonzero source, $j^6 \neq 0$, effectively relaxes this constraint. As a particular example let us consider:

$$j^6 = -\frac{2}{5}e^{-\Phi}\mu\,\mathrm{Re}\Omega\,, \tag{2.10}$$

where μ is an arbitrary real parameter, so that $-\mu$ is proportional to the orientifold/D6-brane tension (μ is positive for orientifolds and negative for D6-branes). The addition of this source term was first considered in [25]. Eq. (2.10) above guarantees that the calibration conditions, which for D6-branes/O6-planes read

$$j^6 \wedge \mathrm{Re}\Omega = 0\,, \qquad j^6 \wedge J = 0\,, \tag{2.11}$$

are satisfied and thus the source wraps supersymmetric cycles. The bound (2.9) should now be replaced by:

$$\mu \geq \frac{5}{16}\left(|\mathcal{W}_2^-|^2 - 3|\mathcal{W}_1^-|^2\right)\,. \tag{2.12}$$

Since μ can be taken to be arbitrary the above equation can always be satisfied, and therefore no longer imposes any constraint on the torsion classes of the manifold.

Let us also note that it is possible to consider the inclusion of more general supersymmetric orientifold six-plane sources, *not* given by eq. (2.10). In this case the second constraint of table 2, i.e. the constraint $d\mathcal{W}_2^- \propto \mathrm{Re}\Omega$, is relaxed. We will still require this source to satisfy the calibration conditions (2.11).

In summary: In the absence of sources the necessary and sufficient conditions for $\mathcal{N}=1$ compactifications with strict SU(3)-structure to four-dimensional AdS_4 space are those listed in table 2. However in the presence of sources the last two of the three constraints may be relaxed. In particular the third constraint can always be relaxed by the addition of orientifold/D6-brane sources of the form (2.10).

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3. Coset spaces and left-invariant SU(3)-structures

In this section we give a brief review of some well-known facts about coset spaces, with special emphasis on the material that will be useful to us in the following (for more extensive reviews see [14, 26, 27]).

Thanks to the uniqueness theorem quoted in appendix A, in dealing with coset spaces of the form G/H it suffices to examine the corresponding algebras $\mathfrak{g}$, $\mathfrak{h}$. Let $\{\mathcal{H}_a\}$ be a basis of generators of the algebra $\mathfrak{h}$, and let $\{\mathcal{K}_i\}$ be a basis of the complement $\mathfrak{k}$ of $\mathfrak{h}$ inside $\mathfrak{g}$, i.e. $a = 1, \ldots, \dim(H)$ and $i = 1, \ldots, \dim(G){-}\dim(H)$. We define the structure constants as follows:

$$
\begin{aligned}
[\mathcal{H}_a, \mathcal{H}_b] &= f^c{}_{ab}\mathcal{H}_c \,, \\
[\mathcal{H}_a, \mathcal{K}_i] &= f^j{}_{ai}\mathcal{K}_j + f^b{}_{ai}\mathcal{H}_b \,, \\
[\mathcal{K}_i, \mathcal{K}_j] &= f^k{}_{ij}\mathcal{K}_k + f^a{}_{ij}\mathcal{H}_a \ .
\end{aligned} \tag{3.1}
$$

If H is connected and semisimple, or compact — as is indeed the case for each H listed in table 1 — one can always find a basis of generators $\{\mathcal{K}_i\}$ such that the structure constants $f^b{}_{ai}$ vanish [26, 28]. In other words: $[\mathcal{H}, \mathcal{K}] \subset \mathcal{K}$, in which case the coset G/H is called *reductive.*

Let x^m, $m = 1, \ldots, \dim(G){-}\dim(H)$, be local coordinates on G/H and let $L(x)$ be a coset representative. The decomposition of the Lie-algebra valued one form

$$
L^{-1}dL = e^i\mathcal{K}_i + \omega^a\mathcal{H}_a \,, \tag{3.2}
$$

defines a coframe $e^i(x)$ on G/H. Moreover, using the commutation relations (3.1), we find

$$
de^i = -\frac{1}{2}f^i{}_{jk}e^j \wedge e^k - f^i{}_{aj}\omega^a \wedge e^j \ . \tag{3.3}
$$

We are interested in forms that are *left-invariant* under the action of G on G/H. One can show that this is the case if and only if for the p-form

$$
\phi = \frac{1}{p!}\phi_{i_1 \ldots i_p} e^{i_1} \wedge \cdots \wedge e^{i_p} \,, \tag{3.4}
$$

its components $\phi_{i_1 \ldots i_p}$ are constants *and*

$$
f^j{}_{a[i_1}\phi_{i_2 \ldots i_p]j} = 0 \ . \tag{3.5}
$$

If we then take the exterior derivative $d\phi$, condition (3.5) ensures that the part coming from the second term in (3.3) drops out so we find that the exterior derivative preserves the left-invariance property. As an aside one can show that harmonic forms must be left-invariant and thus the cohomology of the coset manifold is isomorphic to the cohomology of left-invariant forms.

The strategy we follow in this paper is to restrict ourselves to cosets with left-invariant SU(3)-structure. In other words, we demand that (J, Ω) be left-invariant forms on G/H. From eq. (3.3) it then follows that, given the structure constants of the coset in eq. (3.1), the exterior derivatives $(dJ, d\Omega)$ can be explicitly evaluated. On the other hand, the first

Manifold		b_1	b_2	b_3
SU(2)×SU(2)		0	0	2
$\frac{\mathrm{SU(3)}}{\mathrm{U(1)\times U(1)}}$		0	2	0
$\frac{\mathrm{Sp(2)}}{\mathrm{S(U(2)\times U(1))}}$	max.	0	1	0
	nonmax.	0	1	0
$\frac{\mathrm{G_2}}{\mathrm{SU(3)}}$		0	0	0
$\frac{\mathrm{SU(3)\times U(1)}}{\mathrm{SU(2)}}$		1	0	0
$\frac{\mathrm{SU(2)^2\times U(1)}}{\mathrm{U(1)}}$	$b=0$	1	1	2
	$b\neq 0$	0	0	2
$\frac{\mathrm{SU(3)\times U(1)^2}}{\mathrm{SU(2)\times U(1)}}$	$a=0$	2	2	2
	$a\neq 0$	1	0	0
$\mathrm{SU(2)\times U(1)^3}$		3	3	2
$\frac{\mathrm{SU(2)^2\times U(1)^2}}{\mathrm{U(1)^2}}$	$a=d=0$	2	3	4
	$a\neq 0, d=0$	1	1	2
	$a\neq 0, d\neq 0$	0	0	2
$\frac{\mathrm{SU(2)\times U(1)^4}}{\mathrm{U(1)}}$	$a=0$	4	7	8
	$a\neq 0$	3	3	2
$\frac{\mathrm{SU(2)^3}}{\mathrm{SU(2)}}$		0	0	2
$\frac{\mathrm{SU(2)^3\times U(1)}}{\mathrm{SU(2)\times U(1)}}$		0	0	2

Table 5: Betti numbers of the first, second and third cohomology. For the meaning of the parameters a, b, d see the section on the corresponding coset.

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condition of table 2 is equivalent to the statement that $\mathcal{W}_{1,2}^-$ are the only non-vanishing torsion classes of the coset. As this is not the most general form of $(dJ, d\Omega)$, this condition imposes a constraint on (J, Ω), which may not have any solutions. Provided solutions exist, one can immediately read off the torsion classes $\mathcal{W}_{1,2}^-$. Finally, the second and third conditions of table 2 can be examined to determine whether or not the solutions require the presence of sources.

The procedure described above, when applied to each of the cosets listed in table 1, leads to the results summarized in the introduction. The details of the analysis in each case are presented in section 4.

4. Case by case analysis

In this section we present the details of the analysis for each coset listed in table 1. As explained in section 3, our procedure is as follows: For each coset we first write down the most general left-invariant ansatz for (J, Ω). We then impose the SU(3)-structure conditions (2.2a). In addition, we have to demand that the resulting metric, implicitly defined by (J, Ω), be positive. Next we take into account the structure constants of the coset in order to evaluate $(dJ, d\Omega)$, using eq. (3.3). Finally we impose equations (2.5). In case solutions exist, we read off the torsion classes $\mathcal{W}_{1,2}^-$ and we examine whether or not the Bianchi identity (2.4) requires the presence of sources. The results of this analysis were summarized in the introduction, tables 3 and 4.

Some further remarks about the presentation in the remainder of this section: table 5 displays the Betti numbers of all the cosets under study, providing the reader with a feeling of their topology. Furthermore, in each case we present first the structure constants of the coset. We assume that $\mathfrak{g}$ is generated by $\{E_I\}$, $I = 1, \ldots, \dim(G)$, such that

$$[E_I, E_J] = f^K{}_{IJ} E_K \,, \tag{4.1}$$

and our labelling is such that the E_I with $I = 1, \ldots, 6$ correspond to the $\mathcal{K}_i$ (spanning $\mathfrak{k}$) and the E_I with $I = 7, \ldots, 6 + \dim(H)$ correspond to the $\mathcal{H}_a$ (spanning $\mathfrak{h}$). Then follows the solution (in case it exists) for the SU(3)-structure (J, Ω), expressed in terms of some set of parameters. The conditions on these parameters imposed by the normalization of Ω (eq. (2.2b)) and the positivity of the metric are listed explicitly. We then give the explicit form of the torsion classes $\mathcal{W}_{1,2}^-$.

It is always understood, unless otherwise stated, that each solution satisfies the Bianchi identity (2.4) in the absence of sources. We therefore also explicitly list the condition imposed by the bound (2.9). As explained in section 2, one can always add O6/D6 sources of the form (2.10). The mass parameter m is then no longer determined by (2.9) and thus becomes an extra free parameter — also counted in table 4 — related to the number of sources. Whenever there exist solutions with sources that are not of the form (2.10), it is stated explicitly.

4.1 SU(2) × SU(2)

The structure constants in this case are

$$f^1{}_{23} = f^4{}_{56} = 1 \,, \qquad \text{cyclic} \,. \tag{4.2}$$

In [18] it was shown that there is always a change of basis preserving the form of the structure constants that brings J in diagonal form

$$J = a e^1 \wedge e^4 + b e^2 \wedge e^5 + c e^3 \wedge e^6 \,. \tag{4.3}$$

The most general solution to (2.4), (2.5), (2.6) and (2.9) without sources, $j^6 = 0$, is the nearly-Kähler one:

$$\begin{aligned} J =& a(e^{14} + e^{25} + e^{36}) \,, \\ \Omega =& d \left(e^{156} + e^{426} + e^{453} - e^{126} - e^{153} - e^{423} \right) \\ & - \frac{2id}{\sqrt{3}} \left[e^{123} + e^{456} - \frac{1}{2} \left(e^{156} + e^{426} + e^{453} \right) - \frac{1}{2} \left(e^{423} + e^{153} + e^{126} \right) \right] . \end{aligned} \tag{4.4}$$

with a, the overall scale of the internal geometry, the only free parameter and

$$\begin{aligned}
a &> 0\,, && \text{metric positivity}\,,\\
d^2 &= \frac{2}{\sqrt{3}}a^3\,, && \text{normalization of }\Omega\,,\\
c_1 &:= -\frac{3i}{2}\mathcal{W}_1^- = \frac{a}{d}\,, && \\
\mathcal{W}_2^- &= 0\,, && \\
e^{2\Phi}m^2 &= \frac{5}{12}c_1^2\,. &&
\end{aligned} \tag{4.5}$$

A different solution is possible with a source not proportional to $\mathrm{Re}\Omega$. We have then

$$\begin{aligned}
J =& ae^{14} + be^{25} + ce^{36}\,,\\
\Omega =& -\frac{1}{c_1}\Bigg\{a(e^{234}-e^{156}) + b(e^{246}-e^{135}) + c(e^{126}-e^{345})\\
&\quad -\frac{i}{h}\Big[-2abc(e^{123}+e^{456}) + a(b^2+c^2-a^2)(e^{234}+e^{156}) + b(a^2+c^2-b^2)(e^{153}+e^{426})\\
&\quad + c(a^2+b^2-c^2)(e^{345}+e^{126})\Big]\Bigg\}\,,
\end{aligned} \tag{4.6}$$

with a, b and c three free parameters and

$$\begin{aligned}
& abc > 0\,, \qquad \text{metric positivity}\,,\\
& h = \sqrt{2\,a^2b^2 + 2\,b^2c^2 + 2\,a^2c^2 - a^4 - b^4 - c^4}\,,\\
& \quad \text{and thus} \quad 2\,a^2b^2 + 2\,b^2c^2 + 2\,a^2c^2 - a^4 - b^4 - c^4 > 0\,,\\
& c_1^2 = \frac{h}{2abc}\,,\\
& \mathcal{W}_2^- = -\frac{2i}{3hc_1}\left[\frac{(b^2-c^2)^2 + a^2(-2a^2+b^2+c^2)}{bc}e^{14} + \frac{(c^2-a^2)^2 + b^2(-2b^2+c^2+a^2)}{ac}e^{25}\right.\\
& \qquad\qquad \left. + \frac{(a^2-b^2)^2 + c^2(-2c^2+a^2+b^2)}{ab}e^{36}\right].
\end{aligned} \tag{4.7}$$

One can check that $d\mathcal{W}_2^-$ is not proportional to $\mathrm{Re}\Omega$ unless $|a| = |b| = |c|$, which brings us back to the above solution. The source can have total negative or positive tension. In the latter case this geometry can be created with strictly D-brane sources.

4.2 $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$

This space is also known as the flag manifold $\mathbb{F}(1,2;3)$ or the twistor space $\mathrm{Tw}(\mathbb{CP}^2)$.

We choose a basis such that the structure constants of SU(3) are given by

$$f^1{}_{54} = f^1{}_{36} = f^2{}_{46} = f^2{}_{35} = f^3{}_{47} = f^5{}_{76} = \frac{1}{2}\,, \quad f^1{}_{27} = 1\,, \quad f^3{}_{48} = f^5{}_{68} = \frac{\sqrt{3}}{2}\,, \ \text{cyclic}\,. \tag{4.8}$$

These can be obtained from the Gell-Mann structure constants $f_{\mathrm{GM}ijk}$ using the permutation (12456738). The $\mathrm{U}(1) \times \mathrm{U}(1)$ is then generated by E^7 and E^8.

The G-invariant two-forms and three-forms are spanned by

$$\{e^{12}, e^{34}, e^{56}\}, \qquad \{\rho = e^{245} + e^{135} + e^{146} - e^{236}, \hat{\rho} = e^{235} + e^{136} + e^{246} - e^{145}\}, \tag{4.9}$$

respectively, and there are no invariant one-forms. With the two invariant three-forms, one can construct exactly two invariant almost complex structures: J associated to $\rho + i\hat{\rho}$ and $-J$ associated to $\rho - i\hat{\rho}$. Also, with only these two invariant three-forms there is no room for a source not proportional to $\mathrm{Re}\Omega$.

The most general solution is then given by

$$\begin{aligned} J &= -ae^{12} + be^{34} - ce^{56}\,, \\ \Omega &= d\left[(e^{245} + e^{135} + e^{146} - e^{236}) + i(e^{235} + e^{136} + e^{246} - e^{145})\right]\,, \end{aligned} \tag{4.10}$$

with a, b and c three free parameters and

$$\begin{aligned} & a > 0,\, b > 0,\, c > 0\,, && \text{metric positivity}\,, \\ & d^2 = abc, && \text{normalization of } \Omega\,, \\ & c_1 := -\frac{3i}{2}\mathcal{W}_1^- = -\frac{a+b+c}{2d}\,, \\ & \mathcal{W}_2^- = -\frac{2i}{3d}\left[a(2a-b-c)e^{12} + b(a-2b+c)e^{34} + c(-a-b+2c)e^{56}\right]\,, \\ & c_2 := -\frac{1}{8}|\mathcal{W}_2^-|^2 = -\frac{2}{3abc}\left(a^2+b^2+c^2-(ab+ac+bc)\right)\,, \\ & \frac{2}{5}e^{2\Phi}m^2 = c_2 + \frac{1}{6}c_1^2 = \frac{1}{8abc}\left[-5(a^2+b^2+c^2) + 6(ab+ac+bc)\right] \geq 0\,. \end{aligned} \tag{4.11}$$

The nearly-Kähler limit corresponds to $a = b = c$.

We can also make the connection with the results of [12] by defining the complex one-forms

$$e^{z^1} = a^{1/2}\left(-e^2 + ie^1\right)\,, \qquad e^{z^2} = b^{1/2}\left(-e^3 + ie^4\right)\,, \qquad e^{z^3} = c^{1/2}\left(-e^6 + ie^5\right)\,, \tag{4.12}$$

which satisfy[7]

$$d\begin{pmatrix} e^{z^1} \\ e^{z^2} \\ e^{z^3} \end{pmatrix} = \begin{pmatrix} -\alpha & 0|_{2\times 1} \\ 0|_{1\times 2} & \mathrm{Tr}\,\alpha \end{pmatrix}\begin{pmatrix} e^{z^1} \\ e^{z^2} \\ e^{z^3} \end{pmatrix} - \frac{i}{2c^{1/2}}\begin{pmatrix} \left(\frac{a}{b}\right)^{1/2} e^{\bar{z}^2} \wedge e^{\bar{z}^3} \\ \left(\frac{b}{a}\right)^{1/2} e^{\bar{z}^3} \wedge e^{\bar{z}^1} \\ \left(\frac{c}{(ab)^{1/2}}\right) e^{\bar{z}^1} \wedge e^{\bar{z}^2} \end{pmatrix}\,, \tag{4.13}$$

with α the anti-hermitian matrix of one forms

$$\alpha = i\begin{pmatrix} \omega^7 & 0 \\ 0 & -\frac{1}{2}\omega^7 - \frac{\sqrt{3}}{2}\omega^8 \end{pmatrix}\,. \tag{4.14}$$

If $a = b$ these equations take (up to conventions) the form of eq. (3.10) of [12] with $R = -2c^{1/2}$ and $\sigma = c/a$. By having imposed eq. (3.10) therein, we see that the construction of [12] misses the possibility $a \neq b$.

[7] Note that one has now to take into account the second term on the r.h.s. of (3.3) as these complex one-forms are not left-invariant.

4.3 $\frac{\mathrm{Sp}(2)}{\mathbf{S}(\mathrm{U}(2)\times\mathrm{U}(1))}$

Maximal embedding. The algebra sp(2) $\approx$ so(5) is generated by traceless antisymmetric matrices $\{J^{(ij)}|\ i,j=1,\ldots,5\}$ given by

$$\left(J^{(ij)}\right)_{kl}=\delta^i_k\delta^j_l-\delta^i_l\delta^j_k\ . \tag{4.15}$$

These satisfy the following commutation relations:

$$[J^{(ij)},J^{(kl)}]=\frac{1}{2}\left(\delta^{il}J^{(jk)}+\delta^{jk}J^{(il)}-\delta^{jl}J^{(ik)}-\delta^{ik}J^{(jl)}\right)\ . \tag{4.16}$$

The maximal embedding of su(2)$\oplus$u(1) into sp(2) can be realized by taking su(2) $\approx$ so(3) to be generated by $\{J^{(12)},J^{(13)},J^{(23)}\}$ and u(1) $\approx$ so(2) to be generated by $J^{(45)}$. Let us introduce the following notation:

$$\{E_7,E_8,E_9,E_{10}\}:=\{J^{(12)},J^{(13)},J^{(23)},J^{(45)}\}\,, \tag{4.17}$$

and

$$\{E_1,\ldots,E_6\}:=\{J^{(14)},J^{(15)},J^{(24)},J^{(25)},J^{(34)},J^{(35)}\}\ . \tag{4.18}$$

It follows that in this basis the structure constants are totally antisymmetric, with:

$$f^7{}_{89}=f^7{}_{13}=f^7{}_{24}=f^8{}_{15}=f^8{}_{26}=f^9{}_{35}=f^9{}_{46}=f^{10}{}_{12}=f^{10}{}_{34}=f^{10}{}_{56}=-\frac{1}{2}\,, \tag{4.19}$$

being the only nonzero ones. One can check that $[\mathfrak{k},\mathfrak{k}]=\mathfrak{h}$, as expected for a symmetric coset space in the canonical decomposition.

While there is an invariant two-form: $e^{12}+e^{34}+e^{56}$, there are no invariant one- or three-forms, and thus there is no solution.

Nonmaximal embedding. This space is topologically equivalent to $\mathbb{CP}^3$, which can also be viewed as the twistor space Tw(S^4).

The nonmaximal embedding is realized by embedding su(2)$\oplus$u(1) into an su(2)$\oplus$su(2) $\approx$ so(4) subgroup of sp(2). Using the basis (4.15), let so(4) be the subgroup generated by $\{J^{(ij)}|\ i,j=1,\ldots,4\}$. The isomorphism su(2)$\oplus$su(2) $\approx$ so(4) can be realized explicitly by noting that the two su(2) subalgebras are generated by $\{E_i|\ i=5,6,7\}$ and $\{E_i|\ i=8,9,10\}$, where:

$$\begin{aligned}E_{i+4}&:=\frac{1}{2}\varepsilon^{ijk}J^{(jk)}+J^{(i4)},\\ E_{i+7}&:=\frac{1}{2}\varepsilon^{ijk}J^{(jk)}-J^{(i4)},\qquad i=1,2,3\ .\end{aligned} \tag{4.20}$$

The remaining generators are given by $E_i:=\sqrt{2}J^{(i5)},\ \ i=1,\ldots,4$. With the above definitions the structure constants are totally antisymmetric. The nonzero ones are given by:

$$\begin{aligned}f^5{}_{41}=f^5{}_{32}=f^6{}_{13}=f^6{}_{42}=f^7{}_{21}=f^7{}_{43}=f^8{}_{14}=f^8{}_{32}=f^9{}_{13}=f^9{}_{24}=f^{10}{}_{34}=f^{10}{}_{21}=\frac{1}{2}\,,\\ f^7{}_{56}=f^{10}{}_{89}=-1\ .\end{aligned} \tag{4.21}$$

The su(2)⊕u(1) subalgebra is generated by $E_7, \ldots, E_{10}$.

The G-invariant two-forms and three-forms are spanned by

$$\{e^{12}+e^{34}, e^{56}\}\,, \qquad \{\rho = e^{245}-e^{135}-e^{146}-e^{236}, \hat{\rho} = e^{235}+e^{246}+e^{145}-e^{136}\}\,, \quad (4.22)$$

respectively, and there are no invariant one-forms. The source (if present) must be proportional to $\mathrm{Re}\,\Omega$.

The most general solution is then given by

$$\begin{aligned} J &= a(e^{12}+e^{34}) - ce^{56}\,, \\ \Omega &= d\left[(e^{245}-e^{236}-e^{146}-e^{135}) + i(e^{246}+e^{235}+e^{145}-e^{136})\right]\,, \end{aligned} \quad (4.23)$$

with a and c two free parameters and

$$\begin{aligned} a > 0\,, \quad c > 0, & \qquad \text{metric positivity}\,, \\ d^2 = a^2 c, & \qquad \text{normalization of } \Omega\,, \\ c_1 := -\frac{3i}{2}\mathcal{W}_1^- &= \frac{2a+c}{2d}\,, \\ \mathcal{W}_2^- &= -\frac{2i}{3d}\left[a(a-c)(e^{12}+e^{34}) + 2c(a-c)e^{56}\right]\,, \\ c_2 := -\frac{1}{8}|\mathcal{W}_2^-|^2 &= -\frac{2}{3a^2c}(a-c)^2\,, \\ \frac{2}{5}e^{2\Phi}m^2 = c_2 + \frac{1}{6}c_1^2 &= \frac{1}{8a^2c}\left[-4a^2-5c^2+12ac\right] \geq 0\,. \end{aligned} \quad (4.24)$$

Note that if we set $a = b$ in the $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ solution we get the same result as above. The nearly-Kähler limit corresponds to further setting $a = c$.

Again we can make the connection with the results of [12] by defining the complex one-forms

$$e^{z^1} = a^{1/2}\left(e^2 + ie^1\right)\,, \qquad e^{z^2} = a^{1/2}\left(e^4 + ie^3\right)\,, \qquad e^{z^3} = c^{1/2}\left(e^5 + ie^6\right)\,, \quad (4.25)$$

which satisfy

$$d\begin{pmatrix} e^{z^1} \\ e^{z^2} \\ e^{z^3} \end{pmatrix} = \begin{pmatrix} -\alpha & 0|_{2\times1} \\ 0|_{1\times2} & \mathrm{Tr}\,\alpha \end{pmatrix}\begin{pmatrix} e^{z^1} \\ e^{z^2} \\ e^{z^3} \end{pmatrix} + \frac{i}{2c^{1/2}}\begin{pmatrix} e^{\bar{z}^2}\wedge e^{\bar{z}^3} \\ e^{\bar{z}^3}\wedge e^{\bar{z}^1} \\ \left(\frac{c}{a}\right) e^{\bar{z}^1}\wedge e^{\bar{z}^2} \end{pmatrix}\,, \quad (4.26)$$

with α the anti-hermitian matrix of one forms

$$\alpha = \frac{1}{2}\begin{pmatrix} i(\omega^7+\omega^{10}) & -i\omega^8-\omega^9 \\ -i\omega^8+\omega^9 & i(\omega^7-\omega^{10}) \end{pmatrix}\,. \quad (4.27)$$

These equations take (up to conventions) the form of eq. (3.10) of [12], with $R = 2c^{1/2}$ and $\sigma = c/a$.

4.4 $\frac{\mathrm{G}_2}{\mathrm{SU}(3)}$

The G_2 structure constants are given by (see e.g. [29]):

$$\begin{aligned} f^1{}_{63} &= f^1{}_{45} = f^2{}_{53} = f^2{}_{64} = \frac{1}{\sqrt{3}}\,, \\ f^7{}_{36} &= f^7{}_{45} = f^8{}_{53} = f^8{}_{46} = f^9{}_{56} = f^9{}_{34} = f^{10}{}_{16} = f^{10}{}_{52} \\ &= f^{11}{}_{51} = f^{11}{}_{62} = f^{12}{}_{41} = f^{12}{}_{32} = f^{13}{}_{31} = f^{13}{}_{24} = \frac{1}{2}\,, \\ f^{14}{}_{43} &= f^{14}{}_{56} = \frac{1}{2\sqrt{3}}\,, \qquad f^{14}{}_{21} = \frac{1}{\sqrt{3}}\,, \\ f^{i+6}{}_{j+6,k+6} &= f_{\mathrm{GM}ijk}\,, \end{aligned} \tag{4.28}$$

where $E^7, \ldots, E^{14}$ generate the su(3) subalgebra, and $f_{\mathrm{GM}ijk}$ are the Gell-Mann structure constants.

The G-invariant two-forms and three-forms are spanned by

$$\{e^{12} - e^{34} + e^{56}\}\,, \qquad \{\rho = e^{245} - e^{135} - e^{146} - e^{236}, \hat{\rho} = e^{235} + e^{246} + e^{145} - e^{136}\}\,, \tag{4.29}$$

respectively, and there are no invariant one-forms. And again the source (if present) must be proportional to $\mathrm{Re}\,\Omega$.

The most general solution is then given by

$$\begin{aligned} J &= a(e^{12} - e^{34} + e^{56})\,, \\ \Omega &= d\left[(e^{245} + e^{146} + e^{135} - e^{236}) + i(e^{145} - e^{246} - e^{235} - e^{136})\right]\,, \end{aligned} \tag{4.30}$$

with a, the overall scale, the only free parameter and

$$\begin{aligned} a &> 0\,, && \text{metric positivity}\,, \\ d^2 &= a^3, && \text{normalization of } \Omega\,, \\ c_1 &:= -\frac{3i}{2}\mathcal{W}_1^- = -\frac{\sqrt{3}a}{d}\,, \\ \mathcal{W}_2^- &= 0\,, \\ e^{2\Phi}m^2 &= \frac{5}{12}c_1^2\,. \end{aligned} \tag{4.31}$$

We conclude that the only possibility for this coset is the nearly-Kähler geometry.

4.5 $\frac{\mathrm{SU}(3)\times\mathrm{U}(1)}{\mathrm{SU}(2)}$

The most general case corresponds to taking

$$\begin{aligned} &E_i = G_{i+3}, \quad i = 1, \ldots, 5; \quad E_6 = M; \\ &E_7 = G_1; \quad E_8 = G_2; \qquad\quad E_9 = G_3\,, \end{aligned} \tag{4.32}$$

where the G_i's are the Gell-Mann matrices generating su(3), M generates a u(1), and the su(2) subalgebra is generated by E_7, E_8, E_9. It follows that the SU(2) subgroup is embedded entirely inside the SU(3), so that the total space is given by $\frac{\mathrm{SU(3)}}{\mathrm{SU(2)}} \times \mathrm{U}(1) \simeq S^5 \times S^1$. The structure constants are

$$f^7{}_{89}=1, \quad f^7{}_{14}=f^7{}_{32}=f^8{}_{13}=f^8{}_{24}=f^9{}_{12}=f^9{}_{43}=1/2, \quad f^5{}_{12}=f^5{}_{34}=\frac{\sqrt{3}}{2}, \quad \text{cyclic} . \tag{4.33}$$

There is a solution for non-zero source:

$$\begin{aligned} J =& -a(e^{13}-e^{24})+b(e^{14}+e^{23})+ce^{56}\,, \\ \Omega =& -\frac{\sqrt{3}}{2c_1}\Big\{\left[2a(e^{145}+e^{235})+2b(e^{135}-e^{245})+c(e^{126}+e^{346})\right] \\ &\qquad -\frac{i}{\sqrt{a^2+b^2}}\left[ac(e^{146}+e^{236})+bc(e^{136}-e^{246})-2(a^2+b^2)(e^{125}+e^{345})\right]\Big\}\,, \end{aligned} \tag{4.34}$$

with a, b and c three free parameters and

$$\begin{aligned} c>0\,, \quad a^2+b^2 \neq 0\,, & \qquad \text{metric positivity}\,, \\ \frac{1}{(c_1)^2} = \frac{2}{3}\sqrt{a^2+b^2}, & \qquad \text{normalization of } \Omega\,, \\ c_1 := -\frac{3i}{2}\mathcal{W}_1^-\,, & \\ \mathcal{W}_2^- = \frac{i}{2\,c_1\sqrt{a^2+b^2}} & \left[-a(e^{13}-e^{24})+b(e^{14}+e^{23})-2ce^{56}\right]\,, \\ d\mathcal{W}_2^- = -\frac{i\sqrt{3}}{2\,c_1\sqrt{a^2+b^2}} & \left[a(e^{145}+e^{235})+b(e^{135}-e^{245})-c(e^{126}+e^{346})\right]\,, \\ 3|\mathcal{W}_1^-|^2-|\mathcal{W}_2^-|^2 = 0\,. & \end{aligned} \tag{4.35}$$

Note that $d\mathcal{W}_2^-$ is not proportional to $\mathrm{Re}\,\Omega$, hence the source is not of the form (2.10). Interestingly, if we take the part of the source along $\mathrm{Re}\,\Omega$ to be zero, i.e. $j^6 \wedge \mathrm{Im}\,\Omega = 0$, we find from the last equation in (4.35) that $m = 0$. This would amount to a combination of smeared D6-branes and O6-planes such that the total tension is zero. Allowing for negative total tension (more orientifolds), we could have $m > 0$.

4.6 The remaining cosets

We now turn to the remaining cosets of table 1. These will be shown to either be equivalent to one of the previously examined cases, or to support no solution at all. We give some details in each case for the sake of completeness.

4.6.1 $\frac{\mathrm{SU(2)}^2\times \mathrm{U(1)}}{\mathrm{U(1)}}$

The most general case corresponds to taking

$$\begin{aligned} &E_i = L_i, \qquad i=1,2,3; \qquad E_{i+3}=L'_i, \qquad i=1,2; \qquad E_6 = M; \\ &E_7 = L'_3 - aL_3 - bM, \qquad a,b\in\mathbb{R}\,, \end{aligned} \tag{4.36}$$

where $\{L_i\}$, $\{L'_i\}$ each generates an su(2) algebra, M generates a u(1) component, and the u(1) subalgebra is generated by E_7.

For $b \neq 0$, we will show below that the space (with its SU(3)-structure) is equivalent to the SU(2) × SU(2) example of section 4.1. For $b = 0$, we obtain the space $T^{1,1} \times \mathrm{U}(1)$ (see e.g. [30]) – which is topologically $S^3 \times S^2 \times S^1$. On this latter space it is possible to find a type IIB SU(2)-structure solution which is T-dual to the solution on SU(2)×SU(2) of section 4.1.

The structure constants are then given by

$$\begin{aligned} &f^1{}_{23} = f^7{}_{45} = 1, && \text{cyclic,} \\ &f^3{}_{45} = f^2{}_{17} = f^1{}_{72} = a, && f^6{}_{45} = b\ . \end{aligned} \tag{4.37}$$

There is a nearly-Kähler solution for $a = 1$ and $b \neq 0$:

$$\begin{aligned} J =& k_1 \left[\frac{1}{\sqrt{3}}(e^{15} + e^{24}) + k_3 e^{36} \right] , \\ \Omega =& k_2 \left\{ \frac{1}{\sqrt{3}} \left(e^{235} - e^{134}\right) + k_3 \left(e^{126} + e^{456} - b e^{345}\right) \right. \\ & \left. + i \left[\frac{k_3}{\sqrt{3}} \left(e^{456} - e^{126} + 2e^{256} - 2e^{146}\right) + \frac{1}{3}\left(2e^{123} - e^{134} + e^{235} + e^{345}\right) \right] \right\} , \end{aligned} \tag{4.38}$$

with

$$\begin{aligned} k_3 &= \frac{1}{\sqrt{3}b} , \\ k_2^2 &= \frac{2}{3} k_1^3 && \text{normalization of } \Omega , \\ c_1 &:= -\frac{3i}{2} \mathcal{W}_1^- = -\frac{k_1}{k_2} , \\ \mathcal{W}_2^- &= 0 && \text{nearly-Kähler.} \end{aligned} \tag{4.39}$$

One can check that the metric is indeed positive definite for all $b \neq 0$. There are also non nearly-Kähler solutions with source not proportional to $\mathrm{Re}\,\Omega$.

The fact that this coset gives rise to a nearly-Kähler manifold appears contradictory, as the list of all such manifolds in six dimensions is exhausted by the examples in sections 4.1, 4.4 [18]. The resolution of this puzzle is that the example of the present section is in fact equivalent to the one of section 4.1, as we now show.

The structure constants (4.37) correspond to the exterior algebra $de^I = -1/2 f^I{}_{JK} e^J \wedge$

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e^K. The latter is solved explicitly by the following one-forms:[8]

$$
\begin{aligned}
e^1 &= \sin\psi_1 d\theta_1 - \cos\psi_1 \sin\theta_1 d\phi_1 \,, \\
e^2 &= -\cos\psi_1 d\theta_1 - \sin\psi_1 \sin\theta_1 d\phi_1 \,, \\
e^3 &= -d\psi_1 - d\psi_2 - \cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2 \,, \\
e^4 &= \sin\psi_2 d\theta_2 - \cos\psi_2 \sin\theta_2 d\phi_2 \,, \\
e^5 &= -\cos\psi_2 d\theta_2 - \sin\psi_2 \sin\theta_2 d\phi_2 \,, \\
e^6 &= -d\chi - d\psi_2 - \cos\theta_2 d\phi_2 \,, \\
e^7 &= -d\psi_2 - \cos\theta_2 d\phi_2 \,,
\end{aligned}
\tag{4.40}
$$

where we have introduced the seven coordinates χ, $\phi_{1,2}$, $\theta_{1,2}$, $\psi_{1,2}$. A straightforward albeit tedious computation reveals that, when expressed in terms of the coordinates in (4.40), J and Ω in (4.38) depend on $\psi_{1,2}$ solely via the combination $\psi := \psi_1 + \psi_2$. This effectively reduces the coordinate dependence of J and Ω to six variables, implying that they indeed parameterize a six-dimensional manifold. Let us define the one-forms

$$
\{g^a\} := \{e^a\}|_{\psi_1=\psi;\ \psi_2=0}, \qquad a = 1, \ldots, 6; \tag{4.41}
$$

which manifestly depend on the six coordinates χ, ψ, $\phi_{1,2}$, $\theta_{1,2}$. Due to the previous observation, equations (4.38) still hold if we replace e^a by g^a; we will henceforth understand that such a replacement has been performed.

Let us introduce a new set of one-forms $\hat{g}^a$, defined via:

$$
\begin{pmatrix} \hat{g}^1 \\ \hat{g}^2 \end{pmatrix} = R(-\chi) \begin{pmatrix} g^1 \\ g^2 \end{pmatrix}; \qquad \hat{g}^3 = g^3 - g^6; \qquad \begin{pmatrix} \hat{g}^4 \\ \hat{g}^5 \end{pmatrix} = R(\chi) \begin{pmatrix} g^4 \\ g^5 \end{pmatrix}; \qquad \hat{g}^6 = g^6 \,, \tag{4.42}
$$

where

$$
R(\chi) := \begin{pmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{pmatrix} \,. \tag{4.43}
$$

It is now straightforward to check that J and Ω in (4.38) can be expressed solely in terms of the $\hat{g}^a$'s: this can most easily be seen by noting that

$$
\begin{aligned}
\hat{g}^1 \wedge \hat{g}^5 + \hat{g}^2 \wedge \hat{g}^4 &= g^1 \wedge g^5 + g^2 \wedge g^4 \,, \\
\hat{g}^2 \wedge \hat{g}^5 - \hat{g}^1 \wedge \hat{g}^4 &= g^2 \wedge g^5 - g^1 \wedge g^4 \,, \\
\hat{g}^1 \wedge \hat{g}^2 = g^1 \wedge g^2, \qquad & \qquad \hat{g}^4 \wedge \hat{g}^5 = g^4 \wedge g^5 \,.
\end{aligned}
\tag{4.44}
$$

On the other hand, one can check that the $\hat{g}^a$'s obey the su(2)$\oplus$su(2) algebra, $d\hat{g}^a = -1/2\hat{f}^a{}_{bc}\hat{g}^b \wedge \hat{g}^c$, as given by the structure constants $\hat{f}^1{}_{23} = \hat{f}^4{}_{56} = 1$ and cyclic permutations. This concludes the proof of equivalence to the manifold of section 4.1.

[8] In the following we set $a = 1$ for simplicity; we also set $b = 1$, which can be achieved without loss of generality by a rescaling of the one-form e^6.

4.6.2 $\frac{SU(3)\times U(1)^2}{SU(2)\times U(1)}$

The most general possibility is to take

$$\begin{aligned}&E_i = G_{i+3}, \quad i=1,\dots,4; \qquad E_5 = M, \quad E_6 = N;\\ &E_7 = G_1, \quad E_8 = G_2,\\ &E_9 = G_3, \quad E_{10} = G_8 - aM, \quad a\in\mathbb{R}\,,\end{aligned} \tag{4.45}$$

where the Gell-Mann matrices G_i generate the su(3); M, N generate the two u(1)'s; the su(2)$\oplus$u(1) subalgebra is generated by $E_7,\dots,E_{10}$.

This then leads to the following structure constants

$$\begin{aligned}&f^7{}_{89}=1, \quad f^7{}_{14}=f^7{}_{32}=f^8{}_{13}=f^8{}_{24}=f^9{}_{12}=f^9{}_{43}=1/2, \quad f^{10}{}_{12}=f^{10}{}_{34}=\frac{\sqrt{3}}{2}, \quad \text{cyclic},\\ &f^5{}_{12}=f^5{}_{34}=\frac{a\sqrt{3}}{2}\ .\end{aligned} \tag{4.46}$$

No solution.

4.6.3 $SU(2)\times U(1)^3$

No solution.

4.6.4 $\frac{SU(2)^2\times U(1)^2}{U(1)^2}$

The most general case corresponds to taking

$$\begin{aligned}&E_i = L_i, \qquad E_{i+2} = L'_i, \qquad i=1,2; \qquad E_5 = M, \qquad E_6 = N;\\ &E_7 = L_3 - aM; \qquad E_8 = L'_3 - dN, \qquad a,d\in\mathbb{R}\,,\end{aligned} \tag{4.47}$$

where $\{L_i\}$, $\{L'_i\}$ each generates an su(2) algebra, the M, N each generate a u(1) component, and the u(1)$\oplus$u(1) subalgebra is generated by E_7, E_8.

The structure constants are then given by

$$\begin{aligned}&f^7{}_{12} = f^8{}_{34} = 1, \quad \text{cyclic},\\ &f^5{}_{12} = a, \qquad f^6{}_{34} = d\ .\end{aligned} \tag{4.48}$$

No solution.

4.6.5 $\frac{SU(2)\times U(1)^4}{U(1)}$

The most general possibility consists of taking

$$\begin{aligned}&E_i = L_i, \qquad i=1,2; \quad E_{i+2} = M_i, \quad i=1,\dots,4;\\ &E_7 = L_3 - aM_1, \quad a\in\mathbb{R}\,,\end{aligned} \tag{4.49}$$

where the G_i's generate the su(2), the M_i's each generate a u(1), and the u(1) subalgebra is generated by E_7.

The structure constants are

$$f^1{}_{27} = 1, \quad \text{cyclic}, \quad f^3{}_{12} = a\ . \tag{4.50}$$

No solution.

4.6.6 $\frac{\mathrm{SU}(2)^3}{\mathrm{SU}(2)}$

The first possibility corresponds to taking the SU(2) to be diagonally embedded in $\mathrm{SU}(2)^3$. The generators are taken as follows:

$$\begin{aligned} E_i &= L_i, & E_{i+3} &= L'_i\,, \\ E_{i+6} &= L_i + L'_i + L''_i, & i &= 1,2,3\,, \end{aligned} \tag{4.51}$$

where $\{L_i\}$, $\{L'_i\}$, $\{L''_i\}$ generate an su(2) each, and the su(2) subalgebra is generated by E_7, E_8, E_9. The structure constants read:

$$\begin{aligned} f^1{}_{23} &= f^4{}_{56} = f^7{}_{89} = 1, \quad \text{cyclic}, \\ f^6{}_{75} &= -f^5{}_{76} = f^3{}_{72} = -f^2{}_{73} = 1, \\ -f^6{}_{84} &= f^4{}_{86} = -f^3{}_{81} = f^1{}_{83} = 1, \\ f^5{}_{94} &= -f^4{}_{95} = f^2{}_{91} = -f^1{}_{92} = 1\ . \end{aligned} \tag{4.52}$$

Exactly the same nearly-Kähler solution as (4.4)-(4.5) is possible. This coset is equivalent to $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

The other possibility corresponds to taking the SU(2) to be diagonally embedded in the last two SU(2) factors. The corresponding generators are taken as follows:

$$\begin{aligned} E_i &= L_i, \quad E_{i+3} = L'_i\,, \\ E_{i+6} &= L'_i + L''_i, \quad i = 1,2,3\,, \end{aligned} \tag{4.53}$$

where $\{L_i\}$, $\{L'_i\}$, $\{L''_i\}$ generate an su(2) each, and the su(2) subalgebra is generated by E_7, E_8, E_9. The structure constants read:

$$\begin{aligned} & f^1{}_{23} = f^4{}_{56} = f^7{}_{89} = 1, \quad \text{cyclic}, \\ & f^6{}_{75} = -f^5{}_{76} = -f^6{}_{84} = f^4{}_{86} = f^5{}_{94} = -f^4{}_{95} = 1\ . \end{aligned} \tag{4.54}$$

No solution.

4.6.7 $\frac{\mathrm{SU}(2)^3 \times \mathrm{U}(1)}{\mathrm{SU}(2) \times \mathrm{U}(1)}$

Here we take the SU(2) to be diagonally embedded in the last two SU(2) factors. The corresponding generators are taken as follows:

$$\begin{aligned} E_i &= L_i, \quad E_{i+3} = L'_i, \quad E_{10} = L_3 + M, \\ E_{i+6} &= L'_i + L''_i, \quad i = 1,2,3\,, \end{aligned} \tag{4.55}$$

where $\{L_i\}$, $\{L'_i\}$, $\{L''_i\}$ each generate an su(2), M generates a u(1), and the su(2)$\oplus$u(1) subalgebra is generated by $E_7, \ldots, E_{10}$. The structure constants read:

$$\begin{aligned} & f^1{}_{23} = f^4{}_{56} = f^7{}_{89} = 1, \quad \text{cyclic}, \\ & f^6{}_{75} = -f^5{}_{76} = -f^6{}_{84} = f^4{}_{86} = f^5{}_{94} = -f^4{}_{95} = f^2{}_{10,1} = -f^1{}_{10,2} = 1\ . \end{aligned} \tag{4.56}$$

No solution.

4.6.8 $\frac{SU(3)\times SU(2)^2}{SU(3)}$

This equivalent to the example of section 4.1.

5. Interpolations and domain walls

In this section we put forward a simple ansatz in order to construct supersymmetric interpolations and supersymmetric domain walls. Our starting point will be the AdS_4 solutions presented in section 4. We recall that each of these solutions is of the form:

$$ds^2 = ds^2(AdS_4) + ds^2(\mathcal{M}_6)\,, \tag{5.1}$$

where $\mathcal{M}_6$ is a six-dimensional manifold of SU(3)-structure. More specifically, as reviewed in section 2, its intrinsic torsion is contained in the two torsion classes $\mathcal{W}_1^-$ and $\mathcal{W}_2^-$. In other words, $\mathcal{M}_6$ is a special case of a *half-flat* manifold.[9] As is well known, six-dimensional half-flat manifolds $\mathcal{M}_6$ lift via Hitchin flow to seven-dimensional manifolds $\mathcal{M}_7$ of G_2-holonomy [31, 32]:

$$ds^2(\mathcal{M}_7) = dr^2 + g_{mn}(r,y)dy^m dy^n\,, \tag{5.2}$$

where g_{mn} is the r-dependent metric of $\mathcal{M}_6$ compatible with the r-dependent solution (J,Ω) of the Hitchin-flow equations. This construction is reviewed in appendix B, to which the reader is referred for more details.

Interpolations. In the present context of supersymmetric solutions to ten-dimensional supergravity, one would like to construct a physical realization of the Hitchin flow as follows: we expect that the $AdS_4 \times \mathcal{M}_6$ solutions, presented in section 4, can be obtained as near-horizon limits of supergravity solutions with brane sources. Assuming this is indeed the case, one would like to construct ten-dimensional supergravity solutions which interpolate between the 'near-horizon' metric (5.1) and

$$ds^2 = ds^2(\mathbb{R}^{1,2}) + ds^2(\mathcal{M}_7)\,, \tag{5.3}$$

far from the brane sources, where $ds^2(\mathcal{M}_7)$ is the G_2-holonomy metric (5.2).

Domain walls. Alternatively one could form a (infinitely thin) domain wall in four dimensions, by patching together two solutions with different cosmological constants[10] (see figure 2). The solutions are patched along a three-dimensional hypersurface (the wall) across which the fluxes, as well as the first derivative of the metric, are discontinuous. Accordingly, the wall can be viewed as sourced by localized (infinitely thin) branes.

To obtain a solution with a smooth metric, one has to pass from the infinitely-thin wall approximation to a picture where the wall (and therefore the source branes) become 'thick', i.e. acquire a finite extent in the transverse direction.

[9] A generic half-flat manifold has intrinsic torsion contained in $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$.
[10] see [33] for a recent discussion with explicit solutions.

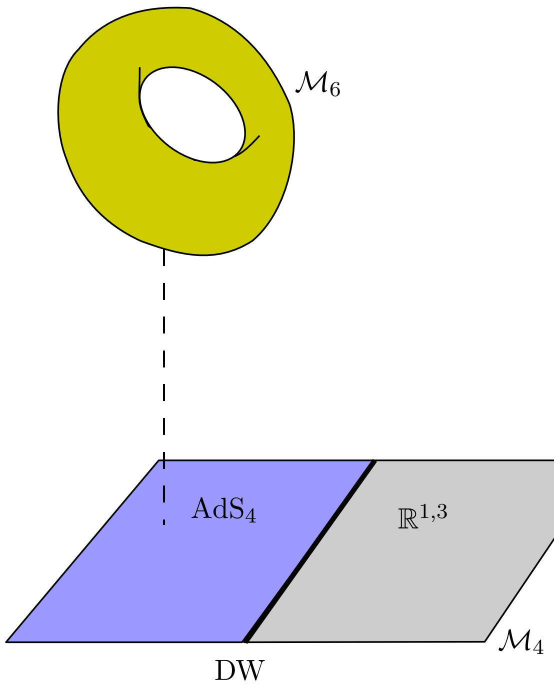

Figure 2: A domain wall in four noncompact dimensions $\mathcal{M}_4$ separating a region of AdS$_4$ from a region of $\mathbb{R}^{1,3}$. The internal manifold $\mathcal{M}_6$ is fibered over $\mathcal{M}_4$. Far from the wall $\mathcal{M}_6$ should be independent of r, the distance from the wall.

Universal ansatz. Motivated by the symmetries of the physical problem, we will here take the metric to be of the form:

$$ds^2 = e^{2A(r)}\left(ds^2(\mathbb{R}^{1,2}) + dr^2\right) + g_{mn}(r,y)dy^m dy^n\,, \tag{5.4}$$

where A is a real, r-dependent function. Note that any metric of the form

$$e^{2U(r)}ds^2(\mathbb{R}^{1,2}) + e^{2V(r)}dr^2\,, \tag{5.5}$$

can be rewritten, by a suitable coordinate transformation $r \to F(r)$, as the flat metric of $\mathbb{R}^{1,3}$, up to an r-dependent conformal factor and is thus included in the above ansatz. To render the problem tractable we will impose a further simplification. Namely we assume that the internal metric is of the form:

$$g_{mn}(r,y) = \omega^2(r)g_{mn}(y)\,, \tag{5.6}$$

for some r-dependent function ω, and we can take $\mathcal{M}_6$ to be any one of the six-dimensional cosets listed in table 3. With this metric ansatz we will be able to treat both interpolating supersymmetric solutions and supersymmetric domain walls simultaneously. The two cases differ only in their asymptotics:

$$\text{Interpolation}: \qquad \omega(r) = \begin{cases} \text{const} & r \to 0 \\ \text{const} \times r & r \to r_\infty \end{cases} \tag{5.7}$$

where $r \to 0$, $r \to r_\infty$ is the near-horizon, far-from-the-source limit respectively, and

$$\text{Domain Wall}: \qquad \omega(r) = \text{const} \quad r \to r_{\pm\infty} \tag{5.8}$$

where $r \to r_{\pm\infty}$ is the limit far from the domain wall, on either side of the wall. Note that in the case of interpolations, in the $r \to r_\infty$ limit, the ten-dimensional space-time asymptotes $\mathbb{R}^{1,3} \times \mathcal{M}_7$, where the metric $ds^2(\mathcal{M}_7)$ (cf. (5.2)) is a cone over $\mathcal{M}_6$. As explained in appendix B, for $\mathcal{M}_7$ to have G_2-holonomy $ds^2(\mathcal{M}_6)$ has to be nearly-Kähler. This is of course possible for all six-dimensional cosets listed in table 3.

5.1 Supersymmetry

In this section we will formulate and solve the equations following from imposing the condition of $\mathcal{N} = 1$ supersymmetry in three dimensions (two real supercharges). At the asymptotic limits of the solution, supersymmetry is enhanced to $\mathcal{N} = 1$ in four dimensions (four real supercharges).

Let us now describe the ansatz of the solution. The analysis is a straightforward generalization of the calculation in [9], except we use here the conventions of [19] (see footnote 6). The spin connection can be read off of eq. (5.4):

$$\begin{aligned} &\nabla_\mu = \partial_\mu + \frac{1}{2}A'\Gamma_\mu\Gamma^r\,, \qquad \mu = 0,1,2\,; \qquad \nabla_r = \partial_r;\\ &\nabla_m = \overset{\circ}{\nabla}_m + \frac{1}{4}g'_{mn}\Gamma^n\Gamma^r\,, \quad m = 4,\dots,9\,; \qquad \overset{\circ}{\nabla}_m := \partial_m + \frac{1}{4}\omega_{mnl}\Gamma^{nl}\,, \end{aligned} \tag{5.9}$$

where the primes denote differentiation by r. In deriving the above we have imposed the following gauge on the vielbein of the internal metric:

$$e'_n{}^a = h_n{}^m e_m{}^a, \qquad h_{mn} := \frac{1}{2}g'_{mn}\,, \tag{5.10}$$

as in [34]. Moreover, we will assume that the r-dependence of the internal metric is such that

$$g'_{mn} = \frac{2\omega'(r)}{\omega(r)}g_{mn}\,, \tag{5.11}$$

for an r-dependent function ω. This will be the case if the vielbein is of the form

$$e_m{}^a(r) = \omega(r)e_m{}^a(r_0)\,, \tag{5.12}$$

which also automatically satisfies the gauge (5.10). A priori, the NSNS three-form as well as the RR forms need only preserve three-dimensional Poincaré invariance so they take the form

$$\begin{aligned} &F_l = \text{vol}_3 \wedge \left(e^A dr \wedge \tilde{F}_{l-4} + \tilde{F}_{3\text{d},l-3}\right) + \hat{F}_l + e^A dr \wedge \hat{F}_{r,l-1}\,, \qquad l = 0,2,4\,;\\ &H = \hat{H} + H_{3\text{d}} + e^A dr \wedge H_r\,. \end{aligned} \tag{5.13}$$

However, we will set

$$\tilde{F}_{3\text{d},l} = \hat{F}_{r,l} = H_{3\text{d}} = H_r = 0\,. \tag{5.14}$$

Note that the domain wall solutions found in [33] as backgrounds generated by brane configurations (before their near-horizon limit is taken) satisfy assumption (5.14), but not (5.6). Let us nevertheless investigate how far we can get by imposing both these conditions.

We make the standard SU(3)-structure ansatz for our ten-dimensional spinor

$$\epsilon = (a\zeta_+ \otimes \eta_+ + a^*\zeta_- \otimes \eta_-) + (b^*\zeta_+ \otimes \eta_- + b\zeta_- \otimes \eta_+)\,, \tag{5.15}$$

where the complex functions a, b and the internal unit spinor η are a priori allowed to be r-dependent. The ten-dimensional gamma-matrices decompose correspondingly as:

$$\begin{aligned} \Gamma^\mu &= \gamma^\mu \otimes 1, \qquad && \mu = 0,\dots,3; \\ \Gamma^m &= \gamma_5 \otimes \gamma^m, \qquad && m = 4,\dots,9\ . \end{aligned} \tag{5.16}$$

Furthermore, we impose the following projection on the four-dimensional spinor ζ

$$\zeta_+ = e^{i\theta} e^{-A} \gamma_r \zeta_-\,, \tag{5.17}$$

which reduces, in general, the supersymmetry of the ansatz from four to two real supercharges. The exponential factor is due to the inverse vielbein, used to convert the curved index on the gamma matrix to a flat one. In the AdS limit $e^{-i\theta}$ becomes the phase of W defined in (2.3). In fact, we can always reabsorb this phase into a redefinition of $\zeta_\pm$ in (5.17) and subsequently in b/a in (5.15). Indeed, it will only ever appear in the combination $e^{i\chi} = (b/a)e^{-i\theta}$.

With the above assumptions, we are ready to proceed to the analysis of the supersymmetry equations, i.e. the vanishing of the gravitino and dilatino variations.[11] After a lengthy but straightforward calculation we find that the solution takes the following form:

$$\begin{aligned} e^{i\chi} &= (b/a)e^{-i\theta} = \text{const}; \qquad |a| = |b| = \text{const} \times e^{\frac{1}{2}A}; \quad \partial_r \eta_\pm = 0\,; \\ H^{(0)} &= -e^{-A} \cos\chi \left(2A' - \Phi' + 3\frac{\omega'}{\omega}\right)\,, \\ m &= -e^{-A-\Phi} \cos\chi \left(5A' - 3\Phi' + 6\frac{\omega'}{\omega}\right)\,, \\ f &= e^{-A-\Phi} \sin\chi \left(3A' - \Phi'\right)\,, \\ F_2^{(0)} &= e^{-A-\Phi} \sin\chi \left(\frac{1}{3}A' + \frac{1}{3}\Phi'\right)\,, \\ F_4^{(0)} &= -e^{-A-\Phi} \cos\chi \left(3A' - \Phi' + 2\frac{\omega'}{\omega}\right)\,, \\ \mathcal{W}_1^- &= \frac{2i}{3} H^{(0)} \tan\chi; \qquad \mathcal{W}_2^- = -ie^{\Phi} F_2'\,, \end{aligned} \tag{5.18}$$

[11] As an alternative to studying the gravitino and dilatino variations directly, it is possible to obtain such domain-wall or interpolating solutions as considered here using the polyform differential equation of appendix A of [35] — which generalizes the pure spinor equations for four-dimensional compactifications found in [5]. This approach will be pursued elsewhere [36].

where in form notation we have:

$$
\begin{aligned}
H &= H^{(0)}\mathrm{Re}\Omega\,,\\
F_2 &= F_2^{(0)}J + F_2'\,,\\
F_4 &= f\mathrm{vol}_4 + \frac{1}{2}F_4^{(0)}J\wedge J\ .
\end{aligned}
\tag{5.19}
$$

Note that we are allowing the mass parameter m to be a function of r, in order to allow for the presence of D8-brane sources.

As a consequence of (5.12), J scales as $\omega^2(r)$, while Ω scales as $\omega^3(r)$. Taking the equations (2.5) into account, it follows that $\mathcal{W}_1^-$ scales as $1/\omega(r)$, while $\mathcal{W}_2^-$ scales as $\omega(r)$. Comparing with the last line of (5.18), we arrive at the following equations:

$$
\begin{aligned}
H^{(0)} &= h\,\frac{1}{\omega(r)}\,,\\
F_2' &= f_2'\,\omega(r)e^{-\Phi}\,,
\end{aligned}
\tag{5.20}
$$

where h and f_2' are r-independent. From (5.18), taking (5.20) into account, we arrive at the following constraint:

$$
\omega e^{-A}\left(\frac{\omega'}{\omega}+\frac{2}{3}A'-\frac{1}{3}\Phi'\right)=\mathrm{const}\,,
\tag{5.21}
$$

where the constant on the right-hand side is equal to $-h/3\cos\chi$.

The AdS_4 limit of the above equations corresponds to Φ, $\omega=\mathrm{const}$, $e^A=R/r$. Indeed upon setting Φ', $\omega'=0$, the reader can verify that eqs. (5.18) reduce precisely to the solution (2.1), provided we identify:

$$
W = e^{-i\theta}A'e^{-A}\,.
\tag{5.22}
$$

From the Bianchi identities of the form fields we find that the configuration generically has sources described by a current j that has an r-index. These are indeed domain wall sources. We should still require that these satisfy appropriate calibration conditions [23, 35]. It is not so difficult to check that if (5.24) holds, this is automatic for the solution of (5.19).

In summary: The solution to the supersymmetry equations is given by eqs. (5.18), supplemented by the constraint eq. (5.21), where the form fields are given by eqs. (5.19) and (5.20). It is also straightforward to check that requiring that the Bianchi identities be solved without such sources, reduces to the AdS_4 solutions of [9].

5.2 Explicit profiles

The solution to the supersymmetry equations of section 5.1 does not uniquely specify the profiles for the warp factors A, ω and the dilaton Φ: given a profile for two of these, the constraint (5.21) can be solved for the third, while (5.18) merely solves for all remaining fields in terms of A, ω and Φ.

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Interpolations. Here we will allow for the presence of general (calibrated) sources, so that the sourceless Bianchi identities (and form-field equations of motion) are violated. The solution to the supersymmetry equation still allows for considerable freedom in the choice of sources. For concreteness we will present a specific solution corresponding to constant dilaton and the following profile for the warp factor:

$$\Phi = \mathrm{const}\,; \qquad e^{A} = 1 + \frac{1}{r}\ . \tag{5.23}$$

Eq. (5.23) ensures that the noncompact space interpolates between AdS_4 in the $r \to 0$ limit and $\mathbb{R}^{1,3}$ in the $r \to \infty$ limit, as follows from the ten-dimensional metric ansatz (5.4).

We will also assume in addition that the internal six-dimensional space is nearly-Kähler, i.e. $\mathcal{W}_2^- = 0$, so that:

$$F_2' = 0\,, \tag{5.24}$$

as follows from (5.18). As explained in appendix B, this allows us to integrate the Hitchin flow equations, so that the six-dimensional nearly-Kähler manifold $\mathcal{M}_6$ can be lifted to a seven-dimensional cone $\mathcal{M}_7$ of G_2 holonomy:

$$ds^2(\mathcal{M}_7) = dr^2 + r^2 g_{mn}(y) dy^m dy^n\ . \tag{5.25}$$

Comparing with (5.4) and (5.6), we see that in order for the ten-dimensional metric to interpolate between $\mathrm{AdS}_4 \times \mathcal{M}_6$ in the $r \to 0$ limit and $\mathbb{R}^{1,2} \times \mathcal{M}_7$ in the $r \to \infty$ limit, we should impose the following asymptotics on ω:

$$\omega(r) = \begin{cases} \mathrm{const} & r \to 0 \\ \mathrm{const} \times r & r \to \infty \end{cases}\ . \tag{5.26}$$

It remains to solve the constraint (5.21). To that end, note that the latter can be rewritten as:

$$\omega(r) = -\frac{h}{3\cos\chi} e^{-2A(r)/3 + \Phi(r)/3} \int^r ds\ e^{5A(s)/3 - \Phi(s)/3}\ . \tag{5.27}$$

Taking (5.23) into account, this can be integrated to give:

$$\omega(r) = -\frac{h}{3\cos\chi} \left\{ 1 + r - \frac{5}{2}(1+r)^{-2/3} {}_2F_1\left(-\frac{2}{3}, -\frac{2}{3}; \frac{1}{3}; -r\right) \right\}\,, \tag{5.28}$$

where the integration constant was determined by imposing the asymptotics (5.26) for small r. The hypergeometric function on the right-hand side above admits an absolutely convergent Taylor-series expansion for $r \leq 1$ (see e.g. [37], §§ 9.100-9.102):

$${}_2F_1\left(-\frac{2}{3}, -\frac{2}{3}; \frac{1}{3}; -r\right) = 1 - \frac{4}{3} r + \mathcal{O}(r^2)\ . \tag{5.29}$$

To analytically continue to $r > 1$ ([37] §§ 9.154-9.155) one uses the identity:

$${}_2F_1\left(-\frac{2}{3}, -\frac{2}{3}; \frac{1}{3}; -r\right) = (1+r)^{5/3} {}_2F_1\left(1, 1; \frac{1}{3}; -r\right)\,, \tag{5.30}$$

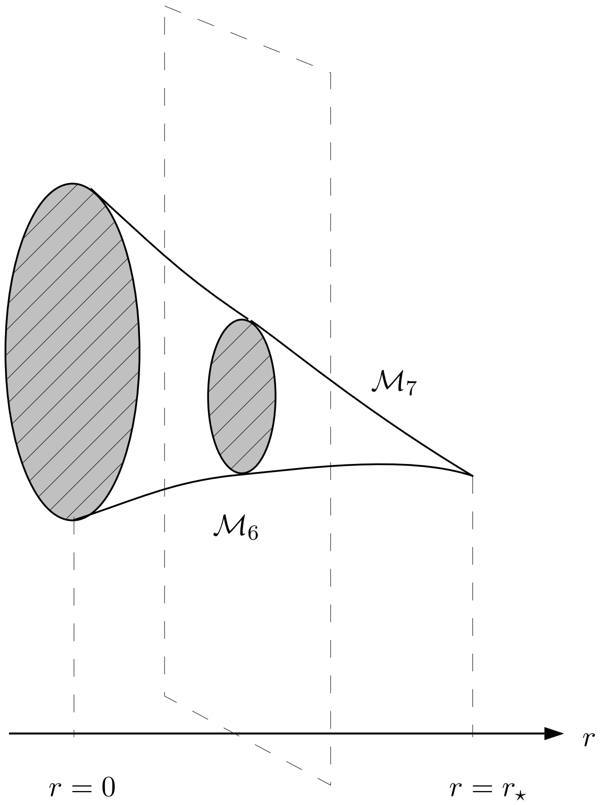

Figure 3: A singular interpolating solution. The internal six-dimensional manifold $\mathcal{M}_6$ is fibered over the radial r-dimension, forming a seven-dimensional manifold $\mathcal{M}_7$ whose r=constant slices are diffeomorphic to $\mathcal{M}_6$. At $r = r_\star$ the six-dimensional fiber shrinks to zero size.

together with the fact that the hypergeometric function on the right-hand side above admits a series expansion of the form:

$$ {}_2F_1\left(1, 1; \frac{1}{3}; -r\right) \sim \frac{1}{r}\log r + \mathcal{O}\left(\frac{1}{r^2}\right) \ . \tag{5.31} $$

From the above discussion we see that (5.28) indeed satisfies the asymptotics (5.26). However note that $\omega \to h/2\cos\chi$ as $r \to 0$ and $\omega \to -rh/3\cos\chi$ as $r \to \infty$, implying that there is an $r_\star \in (0, \infty)$ such that $\omega(r_\star) = 0$.[12] These asymptotic values for $\omega(r)$ are in fact valid for any solution for which the profiles for A, Φ obey the same asymptotics as (5.23). Also in the case where $h = 0$, we see from (5.28) that ω vanishes as $r \to 0$. We conclude that: *for any profile for A, Φ with the same asymptotics as (5.23), the warp factor $\omega(r)$ has a zero at finite radius.*

[12]Numerical analysis shows that $r_\star \simeq 0.293$.

Plugging eqs. (5.23), (5.24), (5.28) into (5.18), allows us to solve for all remaining fields:

$$
\begin{aligned}
f &= -3(1+r)^{-2}\sin\chi\,, \\
m &= 5(1+r)^{-2}\cos\chi + \frac{2h}{\omega}\,, \\
F_2^{(0)} &= -\frac{1}{3}(1+r)^{-2}\sin\chi\,, \\
H^{(0)} &= h\omega^{-1}\,, \\
F_4^{(0)} &= 3(1+r)^{-2}\cos\chi + \frac{2h}{3\omega}\,,
\end{aligned}
\tag{5.32}
$$

where we have set $\Phi = 0$ for simplicity. In particular it follows from the above equations that the Romans mass blows up in the limit $r \to r_\star$. Moreover, in the $r \to \infty$ limit, the NS5 sources also blow up. Indeed, the profile of the NS5-brane sources can be read off of the Bianchi identity for the three-form:

$$
dH = j^5\,, \tag{5.33}
$$

Using (5.32) and (5.19) it follows that j^5 blows up at infinite radius. We conclude that the large-radius behaviour of the solution is unphysical (see figure 3). Nevertheless, as we will show in the following, it is possible to obtain a smooth solution interpolating between AdS_4 vacua of different radii.

Domain walls

Let us now consider a constant dilaton and the following profile for ω:

$$
\Phi = \text{const};\qquad \omega = (2+\tanh r)^{-\frac{2}{5}}\,, \tag{5.34}
$$

which satisfies: $\omega \to$ const, $\omega' \to 0$, as $r \to \pm\infty$. Moreover ω is nowhere-vanishing. The solution therefore has the appropriate asymptotics (5.8) for a domain wall. The limits of ω' ensure that the domain wall sources, i.e. the sources for which j has a component along r, vanish at the endpoints of the radial flow. The constraint (5.21) can be solved in a closed form to obtain:[13]

$$
e^{-A} = \frac{h}{2\cos\chi}(2+\tanh r)^{-\frac{3}{5}}\left[2r + \log(\cosh r)\right]\,, \tag{5.35}
$$

where we have set the integration constant to zero. It can be checked that $e^{-A} \propto r$, as $r \to \pm\infty$, hence the external four-dimensional space asymptotes AdS_4 as $r \to \pm\infty$ (see figure 4).

As another example, let us consider again a constant dilaton and the following profile for the warp factor:

$$
\Phi = \text{const};\qquad e^A = \frac{1}{r} + \frac{1}{r+1}\,. \tag{5.36}
$$

[13]Note that e^{-A} becomes negative for negative r, which amounts to a certain abuse of notation. Equivalently, we could have introduced a different warp factor: $\Delta := e^A$, so that Δ is well-defined for all r.

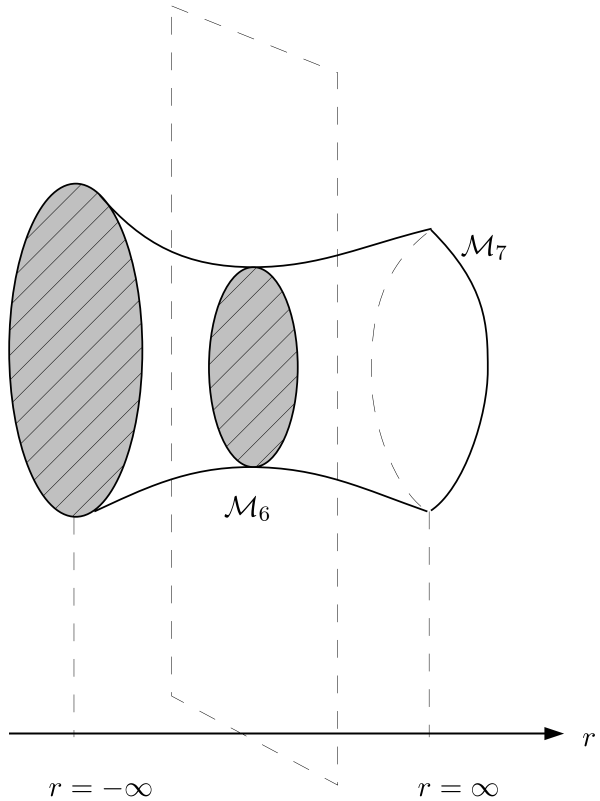

Figure 4: A domain wall solution separating two AdS_4 vacua of different radii. The internal six-dimensional manifold $\mathcal{M}_6$ is fibered over the radial r-dimension, forming a seven-dimensional manifold $\mathcal{M}_7$ whose r=constant slices are diffeomorphic to $\mathcal{M}_6$. As $r \to \pm\infty$ the external four-dimensional space asymptotes to AdS_4.

Eq. (5.36) ensures that the noncompact space interpolates between an AdS_4 space in the $r \to 0$ limit and another AdS_4 space of twice the radius in the $r \to \infty$ limit.

Comparing with (5.4) and (5.6), we see that in order for the ten-dimensional metric to interpolate between two different space-times of the form $AdS_4 \times \mathcal{M}_6$, we should impose the following asymptotics on ω:

$$\omega(r) = \text{const} \qquad r \to 0, \infty\ . \tag{5.37}$$

As in the previous case, the constraint can be solved for ω in a closed form:

$$\omega(r) = \frac{h}{2\cos\chi}\left\{1 - \frac{4r}{1+2r}F_1\left(1, \frac{2}{3}, \frac{1}{3}; \frac{4}{3}; \frac{r}{1+r}, \frac{2r}{1+2r}\right) + C\left(r\frac{1+r}{1+2r}\right)^{2/3}\right\}\ , \tag{5.38}$$

where the generalized hypergeometric function on the right-hand side above is the first of Horn's list (sometimes also called the Appell hypergeometric function of two variables), see e.g. [38], § 5.7.1. The integration constant C can be determined by imposing the asymptotics (5.37) for large r. Indeed, it can be shown that for large r, $F_1 \propto r^{2/3}$.

To arrive at eq. (5.38), we have taken the following identity into account

$$-u^2 \frac{d}{du} f(u) = \left(u + \frac{u}{1+u}\right)^{\frac{5}{3}} , \tag{5.39}$$

where

$$f(u) := -\frac{3}{2} \left\{ \left(u + \frac{u}{1+u}\right)^{\frac{2}{3}} - 4u^{-\frac{1}{3}} F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}; \frac{4}{3}; -\frac{1}{u}, -\frac{2}{u}\right) \right\} . \tag{5.40}$$

Furthermore, using identity (1) of § 5.11 of [38] we have:

$$F_1\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}; \frac{4}{3}; -\frac{1}{u}, -\frac{2}{u}\right) = u(1+u)^{-\frac{2}{3}}(2+u)^{-\frac{1}{3}} F_1\left(1, \frac{2}{3}, \frac{1}{3}; \frac{4}{3}; \frac{1}{1+u}, \frac{2}{2+u}\right) . \tag{5.41}$$

Eq. (5.38) then follows from the above upon setting $r = 1/u$.

This solution, however, has the problem that the domain wall sources blow up at the endpoints of the radial flow.

6. Conclusions

We have reviewed a large class of type IIA $\mathcal{N} = 1$ compactifications to AdS_4, based on left-invariant SU(3)-structures on coset spaces; in the absence of sources they are given in table 3. The moduli spaces of all solutions contain regions corresponding to nearly-Kähler structure, i.e. all cosets of table 3 can be viewed as deformations of nearly-Kähler manifolds, although in the full quantum theory the 'moduli' can only assume discrete values owing to flux quantization. To our knowledge it is an open question whether or not there exist a manifold with non-vanishing torsion classes $\mathcal{W}_{1,2}^-$ with $d\mathcal{W}_2^- \propto \mathrm{Re}\,\Omega$, such that it cannot be deformed to a nearly-Kähler manifold. For that to be the case there would have to be an obstruction to taking the $\mathcal{W}_2^- \to 0$ limit. From the physics point-of-view, this would translate to the statement that the primitive part of the two-form flux has to be non-vanishing.

As we already mentioned in the introduction, the non nearly-Kähler deformation of the $\frac{\mathrm{SU}(3)}{\mathrm{U}(1)\times\mathrm{U}(1)}$ coset was recently analysed in [12], using twistor-space techniques. The solution presented here, however, possesses one more parameter (for a total of three) in addition to the number of parameters in the twistor-space construction of [12]. The remaining cosets of table 3 have also appeared previously under different guises in the literature, starting with the early work of Nilsson and Pope on the Hopf-reduction of the (squashed) seven-sphere [11], and more recently in [10, 12]. Here we have put all these cosets in the same context, and have performed a systematic search for supersymmetric flux compactifications using the tools of (left-invariant) G-structures.

Allowing for (smeared) six-brane/orientifold sources we obtain more possibilities, listed in table 4. These manifolds can serve as starting points for phenomenologically promising compactifications [25]. Given the coset structure of these manifolds, it would certainly be feasible to determine the low-energy physics resulting upon Kaluza-Klein compactification either in a direct way along the lines of [39] or using the supersymmetry to make a suitable ansatz for the expansion forms [13, 40] and construct the superpotential and Kähler potential as in [41]. We leave this interesting line of investigation for future work.

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In the last part of the paper we have obtained smooth interpolations between two AdS_4 vacua of different radii, using the cosets considered here as internal manifolds. These solutions can be interpreted as domain walls in the four noncompact dimensions, and they necessarily contain 'thick' branes. However, we have been unable to obtain physically-sensible profiles of smooth interpolations between $AdS_4 \times \mathcal{M}_6$ and $\mathbb{R}^{1,2} \times \mathcal{M}_7$, where $\mathcal{M}_7$ is the Hitchin lift of $\mathcal{M}_6$. It certainly remains possible that such profiles do exist for more general ansätze than the ones considered here, such as, for example, ansätze for which the form fluxes are allowed to have legs in the radial direction. Another possible generalization is to consider interpolations where the radial evolution of the internal manifold is not simply given by an overall scaling. We hope to return to this issue in the future.

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A. The structure group of coset spaces

In this section we review in some detail the statement that the tangent bundle of the manifold G/H has structure group H.

Let G be a Lie group and H a closed subgroup of G. The group G can be regarded as a principal bundle, denoted by $G(G/H, H)$, with base $M = G/H$ and fibre H. Moreover, the structure group of $G(G/H, H)$ is H ([42], p. 55). The action of G on M induces a map $f : G(G/H, H) \to L(M, S)$, where $L(M, S)$ is the frame bundle of M with structure group $S \subseteq GL(d, \mathbb{R})$, $d := \dim(M)$, and a corresponding map $\varphi : H \to S$. If the action of G is effective (or, equivalently, H contains no nontrivial invariant subgroup of G), both f and φ are isomorphisms ([42], pp. 301-302).[14]

On the other hand, the frame bundle $L(M)$ can be regarded as the associated principal bundle of the tangent bundle of M, $T(M)$. In particular, $T(M)$ and $L(M)$ have the same structure group ([44], pp. 35-36). We conclude, from the discussion in the preceding paragraph, that the structure group of the tangent bundle of $M = G/H$ is isomorphic to H. Note, as a corollary, that by taking $H = \{e\}$ to consist of the identity element of G, it follows that the structure group of the tangent bundle of G, regarded as a manifold, is trivial and the manifold is parallelizable. To summarize:

Let G be a Lie group and H a closed subgroup of G, such that H contains no nontrivial invariant subgroup of G. The structure group of the tangent bundle of $M = G/H$ is H.

[14] It is interesting to note that a similar statement can be formulated for super-coset manifolds: let G be a super-Lie group and H a closed subgroup of G. Then the frame bundle over G/H is equivalent to the principal bundle $G(G/H, H)$ ([43], section 7.2).

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It follows from the above that, as already mentioned in the introduction, it suffices to take H to be isomorphic to SU(3) or a subgroup thereof. It then follows that all possible six-dimensional manifolds M of this type consist of the ones listed in table 1.

Essentially the same list has appeared, for different reasons, in [16]. We would like, however, to make a remark about the entries with H=S(U(2)×U(1)). It is often incorrectly stated in the physics literature, that SU(2) × U(1) is a subgroup of SU(3). To see why this is inaccurate, we first quote the following uniqueness theorem (see e.g. [45], p. 102):

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. There exists a unique connected Lie subgroup H of G, whose Lie algebra is $\mathfrak{h}$.

We will now show that S(U(2)×U(1)) is a subgroup of SU(3), with Lie algebra $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$. First note that S(U(2) × U(1)) is given by

$$\left\{ \begin{pmatrix} e^{i\phi} & 0 \\ 0 & A \end{pmatrix} \,, \quad \text{such that} : A \in \mathrm{U}(2),\ e^{i\phi} \in \mathrm{U}(1),\ e^{i\phi}\mathrm{det}A = 1 \right\}\,, \tag{A.1}$$

and is clearly a subgroup of SU(3). It is also connected, since it is isomorphic to U(2).[15] Indeed, by setting $e^{i\phi} = (\mathrm{det}A)^{-1}$, taking into account the fact that $|\mathrm{det}A| = 1$ for any unitary matrix A, we can therefore identify

$$\mathrm{S}(\mathrm{U}(2)\times\mathrm{U}(1)) \cong \left\{ \begin{pmatrix} (\mathrm{det}A)^{-1} & 0 \\ 0 & A \end{pmatrix} \,, \quad \text{such that} : A \in \mathrm{U}(2) \right\} \cong \mathrm{U}(2)\,. \tag{A.2}$$

Moreover, it is well-known that

$$\mathrm{U}(n) \cong \frac{\mathrm{SU(n)}\times\mathrm{U}(1)}{\mathbb{Z}_n}\,. \tag{A.3}$$

It follows that S(U(2) × U(1))≅ U(2) and SU(2) × U(1) are distinct Lie groups, however they share the same Lie algebra: $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$. From the uniqueness theorem quoted above, it follows that it is S(U(2) × U(1)), but not SU(2) × U(1), that is a subgroup of SU(3).

B. Hitchin flow

Six-dimensional half-flat manifolds lift to seven-dimensional manifolds of G_2-holonomy, as follows [31]: consider $\mathcal{M}_7 = \mathcal{M}_6 \times I$, where $\mathcal{M}_6$ is a six-dimensional half-flat manifold, and I is an interval parameterized by the coordinate r. Moreover, consider the real three-form ϕ defined by:

$$\phi = J \wedge dr + \mathrm{Re}\Omega\,, \tag{B.1}$$

where the SU(3)-structure (J,Ω) of $\mathcal{M}_6$ is now r-dependent. This defines a G_2 structure on $\mathcal{M}_7$. The additional requirement that $\mathcal{M}_7$ have G_2-holonomy is equivalent to the

[15] More generally, one can make the identification S(U(n) × U(1))≅ U(n), upon which one obtains the well-known result that $\mathbb{CP}^n \cong \mathrm{SU}(n+1)/\mathrm{U}(n)$ (see e.g. [46], p. 146).

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requirement that ϕ be closed and coclosed. This is, in its turn, equivalent to the 'Hitchin flow' equations [31]:

$$\begin{aligned} 0 &= \hat{d}J - \partial_r \mathrm{Re}\Omega\,, \\ 0 &= \hat{d}\mathrm{Im}\Omega - J\wedge\partial_r J\,, \end{aligned} \tag{B.2}$$

where $\hat{d}$ is the restriction of the exterior derivative to $\mathcal{M}_6$.

The metric of $\mathcal{M}_7$ is determined by its G_2 structure as follows (see e.g. [47]): define the symmetric two-tensor

$$B_{mn} du^1 \wedge \cdots \wedge du^7 = \left(\frac{\partial}{\partial u^m} \lrcorner\, \phi\right) \wedge \left(\frac{\partial}{\partial u^n} \lrcorner\, \phi\right) \wedge \phi\,, \tag{B.3}$$

where the u^m, $m = 1,\ldots,7$, are local coordinates on $\mathcal{M}_7$. The metric is then given by:

$$g_{mn} = \frac{B_{mn}}{6^{\frac{2}{9}}\det(B)^{\frac{1}{9}}}\,. \tag{B.4}$$

From (B.1) and (B.4) we can read off the metric on $\mathcal{M}_7 = \mathcal{M}_6 \times I$:

$$ds^2(\mathcal{M}_7) = dr^2 + g_{mn}(r,y)dy^m dy^n\,, \tag{B.5}$$

where g_{mn} is the r-dependent metric of $\mathcal{M}_6$ compatible with the r-dependent solution (J,Ω) of the Hitchin-flow equations (B.2).

The Hitchin flow equations can readily be integrated in the case where the six-dimensional manifold space is $\mathcal{M}_6$ is nearly-Kähler, i.e. $\mathcal{W}_2^- = 0$. In this case it can be seen that $\mathcal{M}_7$ is simply a cone over a base $\mathcal{M}_6$. In other words:

$$g_{mn}(r,y)dy^i dy^j = r^2 g_{mn}(r_0,y)dy^i dy^j\,, \tag{B.6}$$

where r_0 is some fixed value of the radial coordinate.

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